

(Co)Simplicial Descent Categories

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Abstract

In this paper we present the notion of *Simplicial descent category*. We propose a set of axioms for a category \mathcal{D} , a class \mathcal{E} of equivalences, and a ‘simple’ functor $\mathbf{s} : \{(\text{co})\text{simplicial objects in } \mathcal{D}\} \rightarrow \mathcal{D}$, ensuring that \mathcal{D} inherits homological and homotopical structure from the category of (co)simplicial objects in \mathcal{D} . They unify the properties satisfied by the following simple functors: (fat) geometric realization for topological spaces; total complex for chain complexes, Navarro’s Thom-Whitney simple for commutative differential graded algebras, Deligne’s simple for mixed Hodge complexes and homotopy limit for fibrant spectra, just to cite a few. Simplicial descent categories are inspired by Guillen-Navarro’s ‘(cubical) descent categories’ [GN]. Our main result is that a cosimplicial descent category \mathcal{D} equipped with a triple provides a Cartan-Eilenberg category structure [GNPR]. As a corollary, we give an existence theorem of right derived functors (in the sense of Quillen).

Introduction

Given a category \mathcal{D} , the category $\Delta^\circ \mathcal{D}$ of simplicial objects in \mathcal{D} supports a great deal of homotopic structure. To illustrate this fact, if $\Delta^\circ \text{Set}_*$ denotes the category of pointed simplicial sets, observe that if \mathcal{D} has coproducts and final object $*$, there is a natural functor

$$\otimes : \Delta^\circ \text{Set}_* \times \Delta^\circ \mathcal{D} \rightarrow \Delta^\circ \mathcal{D} \text{ with } (K \otimes X)_n = * \sqcup \coprod_{K_n - *} X_n \quad (1)$$

This functor induces ‘cofiber sequences’ in $\Delta^\circ \mathcal{D}$, constructed as follows. The suspension ΣX of X is $S^1 \otimes X$, while $\Delta[1] \otimes X$ is the cone of X . Then, the pushout of $X \rightarrow CX$ along a map $f : X \rightarrow Y$ is $C(f)$, the cone of f . This pushout always exists in $\Delta^\circ \mathcal{D}$, and it defines the cofiber sequence

$$X \xrightarrow{f} Y \rightarrow C(f) \rightarrow \Sigma X . \quad (2)$$

On the other hand, one of the fundamental observations of non-abelian homological algebra is that simplicial resolutions are a reasonable generalization to non-abelian settings of resolutions of an object by a chain complex. For instance, cosimplicial Godement resolutions provide resolutions of sheaves of spectra, and of sheaves of complexes with some additional data, as filtrations or multiplicative structures.

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If our aim is to do homological algebra in \mathcal{D} , one approach can be then to transport structure from $\Delta^\circ \mathcal{D}$ to \mathcal{D} , through a sort of ‘simple’ functor $\mathbf{s} : \Delta^\circ \mathcal{D} \rightarrow \mathcal{D}$. Two classical examples are the following. The *geometric realization* $\Delta^\circ \text{Set} \rightarrow \text{Top}$, or more generally $\Delta^\circ \text{Top} \rightarrow \text{Top}$, has been used from the beginning of Algebraic Topology as a tool to develop homotopical and homological structure on Top . In the case of chain complexes over an additive or abelian category, the same can be said about the *simple* functor $\mathbf{s} : \Delta^\circ \text{Ch}_*(\mathcal{A}) \rightarrow \text{Ch}_*(\mathcal{A})$ (that is just the total complex of a double complex). In the cosimplicial setting, *Deligne’s simple* of cosimplicial mixed Hodge complexes [DeIII] is used to define a mixed Hodge structure over the cohomology of singular varieties. Also, Navarro’s *Thom-Whitney simple* [N] of cosimplicial commutative differential graded algebras is a key tool to define the *Thom-Whitney derived functors*.

In the general context of Quillen models, a simplicial model category \mathcal{M} is a simplicial category structure compatible with a model category structure on \mathcal{M} . In this case, there is a satisfactory notion of simple functor $\mathbf{s} : \Delta^\circ \mathcal{M} \rightarrow \mathcal{M}$. It can be made into a Quillen functor through the Reedy model structure on $\Delta^\circ \mathcal{M}$. In [BK], the authors study the homotopy limit $\text{holim} : \Delta \Delta^\circ \text{Set}_* \rightarrow \Delta^\circ \text{Set}_*$. In [T, section 5], the properties of holim are extended to cosimplicial fibrant spectra, in order to study cohomology of sheaves of fibrant spectra.

Returning to the general case of a category \mathcal{D} , equipped with a class of equivalences, and with a simple functor $\mathbf{s} : \Delta^\circ \mathcal{D} \rightarrow \mathcal{D}$, it is natural to ask

Which properties of $\mathbf{s} : \Delta^\circ \mathcal{D} \rightarrow \mathcal{D}$ ensure that \mathcal{D} inherits a suitable homotopic structure from $\Delta^\circ \mathcal{D}$?

More concretely, one would desire that the image under \mathbf{s} of the sequences (2) behave as ‘usual’ cofiber sequences in $\text{Ho}\mathcal{D} = \mathcal{D}[\mathbb{E}^{-1}]$. In particular, they should induce a triangulated category structure on $\text{Ho}\mathcal{D}$ in the stable case. On the other hand, given a functor $F : \mathcal{D} \rightarrow \mathcal{E}$, the expected result is that $\mathbb{L}F$, the (left) derived functor of F , if it exists, could be computed as the simple \mathbf{s} of a suitable simplicial resolution. Dually, for right derived functors one would use a cosimplicial simple $\mathbf{s} : \Delta \mathcal{D} \rightarrow \mathcal{D}$.

In this paper we present the notion of *simplicial descent category* as an answer to these questions. Some of the most remarkable axioms we propose are the following.

COPRODUCTS: \mathcal{D} has finite coproducts. The simple functor commutes with coproducts up to equivalence, and $\mathbb{E} \sqcup \mathbb{E} \subseteq \mathbb{E}$.

EXACTNESS: The simple of a degree-wise equivalence is an equivalence.

ACYCLICITY: The simple of the simplicial cone of a map f is acyclic if and only if $f \in \mathbb{E}$.

EILENBERG-ZILBER: The iterated simple of a bisimplicial object is equivalent to the simple of its diagonal. We explain the axioms in the second section of this paper, after introducing some simplicial preliminaries in the first one.

Simplicial descent categories are inspired by the notion of cubical *homological descent category* introduced in [GN]. However, there are relevant differences between them. For instance, the Eilenberg-Zilber axiom (S4) has no analogue in the cubical setting, since cubical objects have no diagonal.

In order to produce new simplicial descent structures from known ones, we introduce in the third section the *transfer lemma*. The point is to use a functor $\psi : \mathcal{D} \rightarrow \mathcal{D}'$ to transfer a simplicial descent structure from a simplicial descent category \mathcal{D}' to a general category \mathcal{D} . The procedure is motivated by the *singular chain* functor from topological spaces to chain complexes of abelian groups. The singular chain functor is compatible with the (fat) geometric realization and the simple functor \mathbf{s} of chain complexes [Dup, 5.15]. Using this fact, one can transfer the descent structure from chain complexes to topological spaces. The simplicial transfer lemma is similar to the transfer lemma in the cubical setting [GN].

We also prove in sections 4 and 5 that the notion of simplicial descent category includes all examples of ‘simple functors’ cited in the first part of the introduction. The forgetful functor from commutative

differential graded algebras, **Cdga**, to vector spaces is compatible with the Thom-Whitney simple and the simple of cochain complexes [N]. This, together with the transfer lemma, yields a cosimplicial descent structure on **Cdga**. On the other hand, the descent structure on mixed Hodge complexes is indeed obtained from two different simplicial descent structures on filtered cochain complexes, that are related through the ‘decalage’ functor of a filtration. In one of them the equivalences are the filtered quasi-isomorphisms. In the other, they are the maps inducing isomorphism in the second terms of the associated spectral sequences. Finally, the proofs of the axioms of simplicial descent category for fibrant spectra together with the homotopy limit as simple functor is essentially done in [T, section 5]. We also include more examples not mentioned before, as differential graded algebras over a commutative ring, or DG-modules over a fixed DG-category, just to cite a few.

A simplicial descent structure on (\mathcal{D}, E) yields the following properties of $Ho\mathcal{D} = \mathcal{D}[E^{-1}]$. First, $Ho\mathcal{D}$ can be described by means of calculus of left fractions over a quotient category of \mathcal{D} modulo homotopy. Second, the image under the simple functor of the sequences (2) provides cofiber sequences in $Ho\mathcal{D}$ that satisfy the ‘non-stable’ axioms of triangulated categories. When $Ho\mathcal{D}$ is pointed, functor (1) is indeed an action (up to equivalence) of pointed simplicial sets on $\Delta^\circ \mathcal{D}$. In this case, the suspension of any object is a cogroup object in $Ho\mathcal{D}$. Therefore, the following theorem holds

THEOREM [R1, 5.3] *If \mathcal{D} is an stable simplicial descent category -that is, the induced suspension functor is an equivalence of categories- then $Ho\mathcal{D}$ becomes a Verdier’s triangulated structure.*

This triangulated structure is ‘natural’ in the following sense. Assume the suspension functor is an equivalence of categories $\Sigma : \mathcal{D} \rightarrow \mathcal{D}$. Then, for any small category I , $Fun(I, \mathcal{D})$ is again a stable simplicial descent category. Therefore $Fun(I, \mathcal{D})[E^{-1}]$ is triangulated, and $u^* : Fun(J, \mathcal{D})[E^{-1}] \rightarrow Fun(I, \mathcal{D})[E^{-1}]$ is a triangulated functor for each $u : I \rightarrow J$.

(Co)simplicial descent categories are also a suitable setting to derive functors. We treat this question in the last section of this paper. We prove the existence of derived functors (in the sense of Quillen [Q]) with the help of the Cartan-Eilenberg categories machinery [GNPR]. Resolutions are computed through a given triple, compatible with the descent structure. More concretely, if \mathbf{T} is a triple on \mathcal{D} , denote by $\overline{TX} \rightarrow X$ the augmented cosimplicial object produced by \mathbf{T} in the usual way. We prove here the

THEOREM 6.5 *Let $(\mathcal{D}, E, s, \mu, \lambda)$ be a cosimplicial descent category and \mathbf{T} a compatible triple. Consider a functor $F : \mathcal{D} \rightarrow \mathcal{E}$ such that Fs is an isomorphism for every $s \in E$. Set $\mathcal{W} = s\overline{T}^{-1}[E]$. Then there exists a right derived functor $RF : \mathcal{D}[\mathcal{W}^{-1}] \rightarrow \mathcal{E}$, which is a left Kan extension of F . It can be computed as $RF X = Fs\overline{TX}$.*

A deeper study of derived functors in (co)simplicial descent categories -with emphasis on sheaf cohomology of algebras over an operad- will appear in a joint work with A. Roig.

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1 Simplicial Preliminaries

Denote by Δ the *ordinal number category* (or *simplicial category*), with objects the ordered sets $[n] = \{0, \dots, n\}$, $n \geq 0$, and morphisms the order preserving maps. The *face maps* $d^i : [n-1] \rightarrow [n]$ are characterized by $d^i([n-1]) = [n] - \{i\}$, and the degeneracy maps $s^j : [n+1] \rightarrow [n]$ are the surjective monotone maps with $s^j(j) = s^j(j+1)$. They satisfy the well-known *simplicial identities*, and generate all maps in Δ [May].

Note that there is a unique non trivial isomorphism $\Upsilon : \Delta \rightarrow \Delta$, which we call the ‘inverse order’

functor. It is given by $\Upsilon([n]) = [n]$, and if $f : [n] \rightarrow [m]$, then $(\Upsilon f)(i) = m - f(n - i)$. Therefore, $\Upsilon(d^i : [n-1] \rightarrow [n]) = d^{n-i}$ and $\Upsilon(s^j : [n+1] \rightarrow [n]) = s^{n-j}$.

1.1 Simplicial objects

Let \mathcal{D} be a category. A *simplicial object* in \mathcal{D} is a functor $X : \Delta^{op} \rightarrow \mathcal{D}$. It is determined by the data $\{X_n = X([n]), d_i = X(d^i) : X_n \rightarrow X_{n-1}, s_j = X(s^j) : X_n \rightarrow X_{n+1}\}$, where d_i and s_j are called the *face* and *degeneracy* maps, and satisfy the (dual) simplicial identities. A map $f : X \rightarrow Y$ of simplicial objects is a natural transformation of functors, or equivalently a collection of maps $\{f_n : X_n \rightarrow Y_n\}$ of \mathcal{D} compatible with the face and degeneracy maps of X . We denote by $\Delta^\circ \mathcal{D}$ the category of simplicial objects in \mathcal{D} .

Dually, $\Delta \mathcal{D}$ is the category of *cosimplicial objects* in \mathcal{D} . Its objects are functors $X : \Delta \rightarrow \mathcal{D}$, determined by the data $\{X_n = X([n]), d^i = X(d^i) : X_n \rightarrow X_{n-1}, s^j = X(s^j) : X_n \rightarrow X_{n+1}\}$.

We can associate to any object A in \mathcal{D} the *constant simplicial object* $A \times \Delta$, with $(A \times \Delta)_n = A$ for all n , and with identities as face and degeneracy maps. In this way we get the constant functor $- \times \Delta : \mathcal{D} \rightarrow \Delta^\circ \mathcal{D}$ ($- \times \Delta : \mathcal{D} \rightarrow \Delta \mathcal{D}$ in the cosimplicial case). When understood, we will denote also by A the simplicial object $A \times \Delta$.

The ‘inverse order’ functor $\Upsilon : \Delta \rightarrow \Delta$, induces automorphisms $\Delta^\circ \mathcal{D} \rightarrow \Delta^\circ \mathcal{D}$ and $\Delta \mathcal{D} \rightarrow \Delta \mathcal{D}$, which we denote by Υ as well.

The *bisimplicial objects* in \mathcal{D} are the simplicial objects in $\Delta^\circ \mathcal{D}$. They form the category $\Delta^\circ \Delta^\circ \mathcal{D}$. The category $\Delta \Delta \mathcal{D}$ of *bicosimplicial objects* in \mathcal{D} is defined similarly. The *diagonal functor* $D : \Delta^\circ \Delta^\circ \mathcal{D} \rightarrow \Delta^\circ \mathcal{D}$ (or $D : \Delta \Delta \mathcal{D} \rightarrow \Delta \mathcal{D}$) is given by $D(Z_{n,m}) = Z_{n,n}$ (resp. $D(Z^{n,m}) = Z^{n,n}$).

If $X \in \Delta^\circ \mathcal{D}$, the associated bisimplicial objects $X \times \Delta, \Delta \times X \in \Delta^\circ \Delta^\circ \mathcal{D}$ are $(X \times \Delta)_{n,m} = X_n$, $(\Delta \times X)_{n,m} = X_m$. Also $X \in \Delta \mathcal{D}$ induces $X \times \Delta, \Delta \times X \in \Delta \Delta \mathcal{D}$ defined analogously.

Finally, Δ_e denotes the *strict* simplicial category. It is the subcategory of Δ with its same objects, and strict monotone maps as morphisms. As before, we can define the categories $\Delta_e^\circ \mathcal{D}$ and $\Delta_e \mathcal{D}$ of strict simplicial and cosimplicial objects in \mathcal{D} .

Simplicial cones and cylinders

If \mathcal{D} has (finite) coproducts, then simplicial (finite) sets acts on $\Delta^\circ \mathcal{D}$. More concretely, consider $X \in \Delta^\circ \mathcal{D}$ and a simplicial (finite) set K . Then, $K \boxtimes X \in \Delta^\circ \mathcal{D}$ is equal to $\coprod_{K_n} X_n$ in degree n , the coproduct of copies of X_n indexed over K_n . The face and degeneracy maps of $K \boxtimes X$ are obtained by those of K and X .

Recall that $\Delta[k]$ is the simplicial finite set with $\Delta[k]_n = Hom_\Delta([n], [k])$. The cylinder object of $X \in \Delta^\circ \mathcal{D}$ is just $Cyl X = \Delta[1] \boxtimes X$. The maps $d^0, d^1 : [0] \rightarrow [1]$ induce $d_0, d_1 : X \rightarrow Cyl X$, since $X \equiv \Delta[0] \boxtimes X$.

DEFINITION 1.1. If $f : X \rightarrow Y$, and $g : X \rightarrow Z$ are maps in $\Delta^\circ \mathcal{D}$, then their (double mapping) *cylinder*, $Cyl(f, g)$, is given by the pushout [Vo, 2.7]

$$\begin{array}{ccc} X \sqcup X & \xrightarrow{d_0 \sqcup d_1} & Cyl X \\ f \sqcup g \downarrow & & \downarrow \\ Y \sqcup Z & \xrightarrow{I_Y \sqcup J_Z} & Cyl(f, g) \end{array}$$

In degree n , $Cyl(f, g)_n = Y_n \sqcup \coprod^n X_n \sqcup Z_n$. Note that $Cyl(f, g)$, as well as the maps $I_Y : Y \rightarrow Cyl(f, g)$, $J_Z : Z \rightarrow Cyl(f, g)$ (or I, J if Z understood), is a functor on f and g .

Next we introduce a combinatorial lemma, that will be used later. Consider the following commutative diagram in $\Delta^\circ \mathcal{D}$

$$\begin{array}{ccccc} Z' & \xleftarrow{g'} & X' & \xrightarrow{f'} & Y' \\ \alpha \uparrow & & \beta \uparrow & & \gamma \uparrow \\ Z & \xleftarrow{g} & X & \xrightarrow{f} & Y \\ \alpha' \downarrow & & \beta' \downarrow & & \gamma' \downarrow \\ Z'' & \xleftarrow{g''} & X'' & \xrightarrow{f''} & Y'' \end{array}$$

Applying Cyl by rows and columns we obtain

$$Cyl(f', g') \xleftarrow{\delta} Cyl(f, g) \xrightarrow{\delta'} Cyl(f'', g'') \quad Cyl(\alpha', \alpha) \xleftarrow{\hat{g}} Cyl(\beta', \beta) \xrightarrow{\hat{f}} Cyl(\gamma', \gamma)$$

Let $\psi : Cyl(\gamma', \gamma) \rightarrow Cyl(\delta', \delta)$ and $\psi' : Cyl(f'', g'') \rightarrow Cyl(\hat{f}, \hat{g})$ be the respective images under Cyl of:

$$\begin{array}{ccccc} Y' & \xleftarrow{\gamma} & Y & \xrightarrow{\gamma'} & Y'' \\ I \downarrow & & I \downarrow & & I \downarrow \\ Cyl(f', g') & \xleftarrow{\delta} & Cyl(f, g) & \xrightarrow{\delta'} & Cyl(f'', g'') \end{array} \quad ; \quad \begin{array}{ccccc} Z'' & \xleftarrow{g''} & X'' & \xrightarrow{f''} & Y'' \\ I \downarrow & & I \downarrow & & I \downarrow \\ Cyl(\alpha', \alpha) & \xleftarrow{\hat{g}} & Cyl(\beta', \beta) & \xrightarrow{\hat{f}} & Cyl(\gamma', \gamma) \end{array}$$

where each I means the corresponding canonical map.

LEMMA 1.2. *There exists a natural isomorphism of simplicial objects $\Theta : Cyl(\delta', \delta) \rightarrow Cyl(\hat{f}, \hat{g})$, such that $\Theta I = \psi'$ and $\Theta \psi = I$.*

Proof. The proof of the lemma is an easy exercise left to the reader. One can write down the coproducts defining $Cyl(\delta', \delta)_n$ and $Cyl(\hat{f}, \hat{g})_n$ and observe that they are the same after reordering the terms, and this reordering preserves the face and degeneracy maps.

Also, one can deduce the lemma from the invariance of a double pushout by index interchange, and the natural isomorphism $\Delta[1] \boxtimes (K \sqcup L) \simeq (\Delta[1] \boxtimes K) \sqcup (\Delta[1] \boxtimes L)$. \square

Assuming that \mathcal{D} has a final object $*$, the simplicial *cone* of $f : X \rightarrow Y$ in $\Delta^\circ \mathcal{D}$, $C(f)$, is by definition the double mapping cylinder of f and $X \rightarrow *$.

Dually, if \mathcal{D} has (finite) products, $\Delta \mathcal{D}$ has a natural coaction by simplicial (finite) sets. If $X \in \Delta \mathcal{D}$ and K is a simplicial finite set, then X^K is equal to $\prod_{K_n} X^n$ in degree n . Then $X^{\Delta[1]}$ is called the *path* object of X , and again we have natural maps $d^0, d^1 : X^{\Delta[1]} \rightarrow X$.

As before, we can define the (double mapping) path object associated to $f : Y \rightarrow X$ and $g : Z \rightarrow X$, and the dual of lemma 1.2 is satisfied.

In particular, if $f : X \rightarrow Y$ is a map of cosimplicial objects, then the *fibre* $F(f)$ of f is the pullback in $\Delta \mathcal{D}$ of the maps $d^0 \times d^1 : X^{\Delta[1]} \rightarrow X \times X$ and $f \times 0 : X \times 0 \rightarrow X \times X$ (provided 0 is a initial object in \mathcal{D}).

2 Simplicial Descent Categories

Before giving the general axioms of simplicial descent categories, let us illustrate them in the category of chain complexes. Next we recall the definition and basic properties of ‘the simple functor’ from simplicial chain complexes to chain complexes, induced by the total complex of a double complex.

(2.1) Let \mathcal{A} be an additive category and $Ch_*(\mathcal{A})$ be the category of (unbounded) chain complexes in \mathcal{A} . If $X = \{X_n, d_i, s_j\}$ is in $\Delta^\circ Ch_*(\mathcal{A})$, each X_n is a chain complex $\{X_{n,p}, d_{X_n}\}_{p \in \mathbb{Z}}$. Hence X induces a double complex KX with $(KX)_{n,p} = X_{n,p}$. The boundary maps are $d_{X_n} : X_{n,p} \rightarrow X_{n,p-1}$ and $\partial : X_{n,p} \rightarrow X_{n-1,p}$, $\partial = \sum_{i=0}^n (-1)^i d_i$. The *simple* functor $s : \Delta^\circ Ch_*(\mathcal{A}) \rightarrow Ch_*(\mathcal{A})$ of X , sX , is the total complex of KX

$$(sX)_q = \coprod_{p+n=q} X_{n,p} \quad d = \oplus (-1)^p \partial + d_{X_n} : \coprod_{p+n=q} X_{n,p} \longrightarrow \coprod_{p+n=q-1} X_{n,p},$$

In the unbounded case the above coproduct may be not finite. Therefore, we assume the existence of countable sums in \mathcal{A} , assumption that can be dropped in the uniformly bounded below case (for instance the positive chain complexes case). The following properties hold.

- Normalization: Each chain complex $A \in Ch_*(\mathcal{A})$, is canonically a direct summand of $RA = s(A \times \Delta)$, in such a way that the projection $\lambda_A : RA \rightarrow A$ and the inclusion $\rho_A : A \rightarrow RA$ are inverse homotopy equivalences.
- Eilenberg-Zilber [DP, 2.15]: If $Z \in \Delta^\circ \Delta^\circ Ch_*(\mathcal{A})$, the Alexander-Whitney map $\mu_Z : sDZ \rightarrow ssZ$ and the ‘shuffle’ or Eilenberg-Zilber map $\nu_Z : ssZ \rightarrow sDZ$ are inverse homotopy equivalences. In degree n , $(\mu_Z)_n$ is the sum of the maps $Z(d^0 \dots d^j \cdot d^0, d^p d^{p-1} \dots d^{j+1}) : Z_{p,p,q} \rightarrow Z_{i,j,q}$, $i+j=p$, $p+q=n$. The shuffle map is defined in degree n as the sum of the maps

$$\nu_Z(i,j) = \sum_{(\alpha,\beta)} \epsilon(\alpha,\beta) Z(s^{\alpha_j} s^{\alpha_{j-1}} \dots s^{\alpha_1}, s^{\beta_i} s^{\beta_{i-1}} \dots s^{\beta_1}) : Z_{i,j,q} \rightarrow Z_{i+j,i+j,q} \quad i+j+q=n$$

where the sum is indexed over the (i,j) -shuffles (α,β) , and $\epsilon(\alpha,\beta)$ is the sign of (α,β) [EM].

In the abelian case, we also have

- Exactness [B, p. 98, ex. 1] If X is a simplicial object such that X_n is acyclic for all n , then sX is so.
- In addition, if $f : A \rightarrow B$ is a morphism of chain complexes, it is easy to check that the simple of the simplicial cone of f , $sC(f \times \Delta)$, is homotopic to the usual cone of f , $c(f)_n = B_n \oplus A_{n-1}$; $d = d^B + f - d^A$. Then, we deduce the following acyclicity property from the homology long exact sequence associated with f .
- Acyclicity: A map f of chain complexes is a quasi-isomorphism if and only if the simple of its simplicial cone is acyclic.

Next we introduce some notations.

Let \mathcal{C} and \mathcal{D} be two categories, \mathbf{E} a class of morphisms in \mathcal{D} and $Fun(\mathcal{C}, \mathcal{D})$ the category of functors from \mathcal{C} to \mathcal{D} . Denote also by \mathbf{E} the class of morphisms $\tau : F \rightarrow G$ in $Fun(\mathcal{C}, \mathcal{D})$ such that $\tau_C \in \mathbf{E}$ for all $C \in \mathcal{C}$. If $C \in \mathcal{C}$, functor $ev_C : Fun(\mathcal{C}, \mathcal{D}) \rightarrow \mathcal{D}$, $F \mapsto FC$, induces $ev_C : Fun(\mathcal{C}, \mathcal{D})[\mathbf{E}^{-1}] \rightarrow \mathcal{D}[\mathbf{E}^{-1}]$ between the corresponding localized categories. By abuse of notation, we will write $ev_C(\tau) = \tau_C$ if $\tau : F \rightarrow G$ is a morphism in $Fun(\mathcal{C}, \mathcal{D})[\mathbf{E}^{-1}]$.

Recall that the class \mathbf{E} is said to be *saturated* if it holds that $f \in \mathbf{E}$ if and only if f is an isomorphism in $\mathcal{D}[\mathbf{E}^{-1}]$. In particular, \mathbf{E} contains all isomorphisms of \mathcal{D} , is closed by retracts and satisfies the 2-out-of-3 property.

DEFINITION 2.2. A *simplicial descent category* is the data $(\mathcal{D}, \mathbf{E}, s, \mu, \lambda)$ where

- (S1) \mathcal{D} is a category with finite coproducts, initial object 0 and final object $*$.
- (S2) \mathbf{E} is a saturated class of morphisms in \mathcal{D} , stable by coproducts (that is $\mathbf{E} \sqcup \mathbf{E} \subseteq \mathbf{E}$). Morphisms in \mathbf{E} will be called *equivalences*, and an object A such that $A \rightarrow *$ is in \mathbf{E} will be called *acyclic*.
- (S3) $s : \Delta^\circ \mathcal{D} \rightarrow \mathcal{D}$ is a functor, called the *simple functor*, which commutes with coproducts up to equivalence.

That is, the canonical morphism $\mathbf{s}X \sqcup \mathbf{s}Y \rightarrow \mathbf{s}(X \sqcup Y)$ is in \mathbf{E} for all X, Y in $\Delta^\circ \mathcal{D}$.

(S 4) $\mu : \mathbf{s}\mathcal{D} \rightarrow \mathbf{ss}$ is an isomorphism in $Fun(\Delta^\circ \Delta^\circ \mathcal{D}, \mathcal{D})[\mathbf{E}^{-1}]$. If $Z \in \Delta^\circ \Delta^\circ \mathcal{D}$, recall that $\mathbf{s}\mathcal{D}Z$ is the simple of the diagonal of Z . On the other hand $\mathbf{ss}Z = \mathbf{s}(n \rightarrow \mathbf{s}(m \rightarrow Z_{n,m}))$ is the iterated simple of Z .

(S 5) Any X is a ‘weak deformation’ retract of $\mathbf{s}(X \times \Delta)$ in a natural way through λ . That is, $\lambda : \mathbf{s}(- \times \Delta) \rightarrow Id_{\mathcal{D}}$ is a natural transformation with $\lambda_X \in \mathbf{E}$ for all $X \in \mathcal{D}$. In addition, there exists $\rho : Id_{\mathcal{D}} \rightarrow \mathbf{s}(- \times \Delta)$ with $\rho\lambda = Id$. Also, λ is assumed to be compatible with μ in the sense of (2.3) below.

(S 6) If $f : X \rightarrow Y$ is a morphism in $\Delta^\circ \mathcal{D}$ with $f_n \in \mathbf{E}$ for all n , then $\mathbf{s}f \in \mathbf{E}$.

(S 7) If $f : A \rightarrow B$ is a morphism in \mathcal{D} , then $f \in \mathbf{E}$ if and only if $\mathbf{s}C(f \times \Delta)$, the simple of its simplicial cone is acyclic.

(S 8) It holds that $\mathbf{s}\Upsilon f \in \mathbf{E}$ if (and only if) $\mathbf{s}f \in \mathbf{E}$, where $\Upsilon : \Delta^\circ \mathcal{D} \rightarrow \Delta^\circ \mathcal{D}$ is the ‘inverse order’ functor described previously.

(2.3)[Compatibility between λ and μ] Write $R := \mathbf{s}(- \times \Delta) : \mathcal{D} \rightarrow \mathcal{D}$. Given $X \in \Delta^\circ \mathcal{D}$, note that $\mathbf{ss}(X \times \Delta) = \mathbf{s}(n \rightarrow \mathbf{s}(RX_n))$ and $\mathbf{ss}(\Delta \times X) = R\mathbf{s}X$. The compositions

$$\mathbf{s}X \xrightarrow{\mu_{\Delta \times X}} R\mathbf{s}X \xrightarrow{\lambda_{\mathbf{s}X}} \mathbf{s}X \quad \mathbf{s}X \xrightarrow{\mu_{X \times \Delta}} \mathbf{s}(RX) \xrightarrow{\mathbf{s}(\lambda_X)} \mathbf{s}X \quad (3)$$

give rise to isomorphisms of \mathbf{s} in $Fun(\mathcal{D}, \mathcal{D})[\mathbf{E}^{-1}]$. Then, λ is said to be *compatible* with μ if the above isomorphisms are the identity in $Fun(\mathcal{D}, \mathcal{D})[\mathbf{E}^{-1}]$.

An *additive* simplicial descent category is a simplicial descent category which is also additive, and such that the simple functor is an additive functor. Also, μ must be an isomorphism in $Fun_{ad}(\Delta^\circ \Delta^\circ \mathcal{D}, \mathcal{D})[\mathbf{E}^{-1}]$, where $Fun_{ad}(\Delta^\circ \Delta^\circ \mathcal{D}, \mathcal{D})$ is the category of additive functors from $\Delta^\circ \Delta^\circ \mathcal{D}$ to \mathcal{D} .

The opposite of an (additive) simplicial descent category is by definition an (additive) *cosimplicial descent category*.

REMARK 2.4.

0.- The transformation μ of (S 4) is, in our examples, a natural transformation $\mu : \mathbf{s}\mathcal{D} \rightarrow \mathbf{ss}$, or $\mu : \mathbf{ss} \rightarrow \mathbf{s}\mathcal{D}$ in $Fun(\Delta^\circ \Delta^\circ \mathcal{D}, \mathcal{D})$ such that $\mu_Z \in \mathbf{E}$ for all $Z \in \Delta^\circ \Delta^\circ \mathcal{D}$.

1.- Note that (S 4) provides an isomorphism of $Ho\mathcal{D}$ between $\mathbf{s}(n \rightarrow \mathbf{s}(m \rightarrow Z_{n,m}))$ and $\mathbf{s}(m \rightarrow \mathbf{s}(n \rightarrow Z_{n,m}))$, since the diagonals of $\{Z_{n,m}\}$ and $\{Z_{m,n}\}$ are the same.

2.- In many examples of simplicial descent categories, the simple functor is a coend

$$\mathbf{s}X = \int_n X_n \boxtimes C^n \quad (4)$$

where $C \in \Delta \mathcal{D}$ is fixed and \boxtimes is a monoidal structure on \mathcal{D} .

A map from C to the unit element u for \boxtimes provides a natural transformation $\lambda : \mathbf{s}(- \times \Delta) \rightarrow Id$, while a quasi-inverse of λ comes usually from the choice of an isomorphism between C^0 and the unit element u for \boxtimes . More concretely, ρ_A is just the canonical map $A \equiv A \boxtimes C^0 \rightarrow \int_n A \boxtimes C^n$.

On the other hand, a coalgebra structure $C^n \rightarrow C^n \boxtimes C^n$ provides a natural transformation $\mu : \mathbf{s}\mathcal{D} \rightarrow \mathbf{ss}$ compatible with λ .

This is the case of (commutative) differential graded algebras and topological spaces. If R is a commutative ring, the simple functor of chain complexes of R -modules can be written in this way as well.

3.- If \mathcal{D} is an arbitrary category, one can always consider trivial simple functors $\mathbf{s} : \Delta^\circ \mathcal{D} \rightarrow \mathcal{D}$, as $\mathbf{s}X = X_k$ where $k \geq 0$ is fixed. They always provide trivial simplicial descent structures on \mathcal{D} , where the equivalences

are forced to be all morphisms of \mathcal{D} (since **(S 7)** holds). As all maps are equivalences, the associated homotopy category $\mathcal{D}[\mathbf{E}^{-1}]$ is equivalent to the trivial category with one object and one map.

4.- In [R1, 3.6] it is proven that \mathbf{s} sends simplicial homotopy equivalences in $\Delta^\circ \mathcal{D}$ to equivalences in \mathcal{D} . Also, if f is a morphism of $\Delta^\circ \Delta^\circ \mathcal{D}$ such that $\mathbf{s}f_{n,*} \in \mathbf{E}$ for all n , then **(S 4)** and **(S 6)** imply that $\mathbf{s}Df \in \mathbf{E}$. Then, the class $\{f \in \Delta^\circ \mathcal{D} \mid \mathbf{s}f \in \mathbf{E}\}$ is a Δ -closed class in the sense of [Vo].

EXAMPLE 2.5. Let $\mathcal{D} = Ch_*(\mathcal{A})$ be the category of chain complexes over an abelian category \mathcal{A} (with countable sums in the unbounded case). Set \mathbf{W} = quasi-isomorphisms as class of equivalences in \mathcal{D} . Then \mathcal{D}, \mathbf{W} together with the data \mathbf{s}, μ, λ given in (2.1) is a simplicial descent category, that we denote by ${}_Q Ch_*(\mathcal{A})$. The exactness property given there implies **(S 6)**. Indeed, assume $f_n \in \mathbf{E}$ for all n . Then, by **(S 7)**, so is $\mathbf{s}C(f_n) \rightarrow *$, as well as $T = \mathbf{s}(n \rightarrow \mathbf{s}C(f_n)) \rightarrow *$. By the Eilenberg-Zilber property, $T' = \mathbf{s}(n \rightarrow \mathbf{s}(m \rightarrow C(f_m)_n))$ is also acyclic. But \mathbf{s} commutes with direct sums, so T' is isomorphic to $\mathbf{s}C(\mathbf{s}f)$. Thus $\mathbf{s}f \in \mathbf{E}$ by **(S 7)**.

If \mathcal{A} is additive, then $(Ch_*(\mathcal{A}), \lambda, \mu, \rho)$ with the class of equivalences $\mathbf{S} = \{\text{homotopy equivalences}\}$ is also a simplicial descent category. It will be referred to as ${}_H Ch_*(\mathcal{A})$. The axioms can be proved as in the abelian case, except the exactness axiom, whose proof is the same as for first quadrant double complexes (see for instance [B, §5]).

The notion of simplicial descent category is natural in the sense of the following proposition, whose straightforward proof is left to the reader.

PROPOSITION 2.6. *If I is a small category and $(\mathcal{D}, \mathbf{E}_{\mathcal{D}}, \mathbf{s}_{\mathcal{D}}, \lambda_{\mathcal{D}}, \mu_{\mathcal{D}})$ is a simplicial descent category, then the category of functors from I to \mathcal{D} , $I\mathcal{D}$, together with the data $(\mathbf{E}_{I\mathcal{D}}, \mathbf{s}_{I\mathcal{D}}, \lambda_{I\mathcal{D}}, \mu_{I\mathcal{D}})$ defined object-wise is a simplicial descent category. More concretely, if $X : \Delta^{op} \rightarrow I\mathcal{D}$ then $(\mathbf{s}_{I\mathcal{D}}(X))(i) = \mathbf{s}_{\mathcal{D}}(n \rightarrow X_n(i))$. The class of equivalences is $\mathbf{E}_{I\mathcal{D}} = \{f \text{ such that } f(i) \in \mathbf{E}_{\mathcal{D}} \text{ for all } i \in I\}$.*

We finish this section with some formal consequences of the axioms. From now on $(\mathcal{D}, \mathbf{E}, \mathbf{s}, \mu, \lambda)$ denotes a simplicial descent category.

LEMMA 2.7. *Consider a commutative diagram in $\Delta^\circ \mathcal{D}$*

$$\begin{array}{ccccc} Z & \xleftarrow{g} & X & \xrightarrow{f} & Y \\ \alpha \downarrow & & \beta \downarrow & & \downarrow \gamma \\ Z' & \xleftarrow{g'} & X' & \xrightarrow{f'} & Y', \end{array}$$

such that $\mathbf{s}\alpha, \mathbf{s}\beta$ and $\mathbf{s}\gamma$ are equivalences. Then the induced morphism $\mathbf{s}Cyl(f, g) \rightarrow \mathbf{s}Cyl(f', g')$ is also in \mathbf{E} . This is the case, for instance, when α_n, β_n and γ_n are equivalences for all n .

Proof. The case $\alpha_n, \beta_n, \gamma_n \in \mathbf{E}$ for all n follows directly from **(S 2)** and **(S 6)**, and the general case follows from the following result. \square

LEMMA 2.8. *If $f : X \rightarrow Y$ and $g : X \rightarrow Z$ are maps in $\Delta^\circ \mathcal{D}$, then $\mathbf{s}Cyl(f, g)$ is naturally isomorphic to $L = \mathbf{s}C((\mathbf{s}f) \times \Delta, (\mathbf{s}g) \times \Delta)$ in $Ho\mathcal{D}$. This isomorphism commutes with the respective canonical maps from $\mathbf{R}sY, \mathbf{R}sZ, \mathbf{s}Y$, and $\mathbf{s}Z$ to L and $\mathbf{s}Cyl(f, g)$.*

Proof. Note that $\Delta[1] \boxtimes X$ is the diagonal of the bisimplicial object that is equal to $\coprod_{\Delta[1]_n} X_m$ in bidegree (n, m) . Hence $Cyl(f, g) = DT$, where T is the bisimplicial object with $T_{n,m} = Y_m \sqcup \coprod^n X_m \sqcup Z_m$. Then, **(S 4)** provides an isomorphism in $Ho\mathcal{D}$ relating $\mathbf{s}Cyl(f, g)$ to $\mathbf{s}sT$. But for each n , $\mathbf{s}(m \rightarrow T_{n,m})$ is equivalent

to $Cyl((sf) \times \Delta, (sg) \times \Delta)_n$ by **(S 3)**. Therefore, **(S 6)** ensures the existence of $\nu : sCyl(f, g) \xrightarrow{\sim} sCyl((sf) \times \Delta, (sg) \times \Delta)$ in HoD . On the other hand, last assertion follows by definition and by the compatibility between λ and μ . \square

PROPOSITION 2.9. *Given a map $f : X \rightarrow Y$ of simplicial objects, then $sf \in E$ if and only if sCf is acyclic.*

Moreover, if $f : X \rightarrow Y$ and $g : X \rightarrow Y$ are maps in $\Delta^\circ D$ then

- a)** *sf is an equivalence if and only if the simple of $I_Z : Z \rightarrow Cyl(f, g)$ is so.*
- b)** *sg is an equivalence if and only if the simple of $I_Y : Y \rightarrow Cyl(f, g)$ is so.*

Proof. Denote by $\theta : X \rightarrow * \times \Delta$ the trivial map. By **(S 5)**, $s(* \times \Delta) \rightarrow * \in E$, and by lemma 2.7 there is an equivalence between $sC(sf \times \Delta)$ and $L = sCyl(sf \times \Delta, s\theta \times \Delta)$. But, as we have seen, $L \simeq sCyl(f, \theta) = sCf$ in HoD . Hence $sC(sf \times \Delta)$ and sCf are isomorphic in HoD , so **(S 7)** ensures that $sf \in E$ if and only if sCf is acyclic.

To see a) and b), we can assume that f, g are maps in D , since $sCyl(f, g) \equiv sCyl((sf) \times \Delta, (sg) \times \Delta)$ and Rh is an equivalence if and only if h is. To relax the notations we write $h \times \Delta = h$ for a map h of D . It suffices to prove that $f \in E$ if and only if $s(I_Z : Z \rightarrow Cyl(f, g))$ is so. Indeed, it is easy to check that $\Upsilon(Cyl(f, g)) \equiv Cyl(g, f)$ if f, g are constant maps. Then, b) can be formally deduced from a) using **(S 8)**. To prove a), consider the diagram

$$\begin{array}{ccccc} * & \xleftarrow{\quad} & 0 & \xrightarrow{\quad} & 0 \\ \uparrow & & \uparrow & & \uparrow \\ Z & \xleftarrow{\quad} & 0 & \xrightarrow{\quad} & 0 \\ Id_Z \downarrow & & \downarrow & & \downarrow \\ Z & \xleftarrow{\quad g \quad} & X & \xrightarrow{\quad f \quad} & Y \end{array}$$

By lemma 1.2 we get an isomorphism in $\Delta^\circ D$ between the cone of $I_Z : Z \rightarrow Cyl(f, g)$, and $Cyl(f, \alpha)$. Here α is the map from X to CZ induced by g , and CZ is the cone of $Id_Z : Z \rightarrow Z$. By **(S 7)**, sCZ is acyclic, so $s\alpha$ is equivalent to the trivial map $X \rightarrow *$. Then, it follows from lemma 2.7 that $sCyl(sf, s\alpha)$ is isomorphic in HoD to $sC(sf) \simeq sCf$. We conclude that sCf is isomorphic in HoD to sCI_Z , then the result follows from the first assertion. \square

3 Descent functors and the transfer lemma

DEFINITION 3.1. If (D, s, E, μ, λ) and $(D', E', s', \mu', \lambda')$ are simplicial descent categories, a *descent functor* between them is a functor $\psi : D \rightarrow D'$ such that

(DF 0) ψ preserves equivalences, that is, $\psi(E) \subseteq E'$.

(DF 1) The canonical map $\psi(X) \sqcup \psi(Y) \rightarrow \psi(X \sqcup Y)$ is in E' for each $X, Y \in D$.

(DF 2) Consider the diagram

$$\begin{array}{ccc} \Delta^\circ D & \xrightarrow{\psi} & \Delta^\circ D' \\ s \downarrow & & \downarrow s' \\ D & \xrightarrow{\psi} & D' \end{array} \quad (5)$$

There exists an isomorphism $\Theta : \psi s \rightarrow s' \psi$ of $Fun(\Delta^\circ D, D')[E'^{-1}]$. In addition Θ must be compatible with λ, λ' and with μ, μ' in the following sense:

- I. If $X \in \mathcal{D}$, then $\lambda'_{\psi(X)} \Theta_{X \times \Delta} = \psi(\lambda_X)$ in $\text{Fun}(\mathcal{D}, \mathcal{D}')[E'^{-1}]$.
 II. If $Z \in \Delta^\circ \Delta^\circ \mathcal{D}$, then the following diagram

$$\begin{array}{ccc}
 \psi s D Z & \xrightarrow{\Theta_{DZ}} & s' D(\psi Z) \\
 \psi(\mu_Z) \downarrow & & \downarrow \mu'_{\psi Z} \\
 \psi s s Z & \xrightarrow{\Theta_{sZ}} s' \psi(s Z) \xrightarrow{s'(\Theta_Z)} & s' s'(\psi Z)
 \end{array}$$

commutes in $\text{Fun}(\Delta^\circ \Delta^\circ \mathcal{D}, \mathcal{D}')[E'^{-1}]$.

EXAMPLE 3.2. 1.- If we replace the Alexander-Whitney map μ by the Eilenberg-Zilber map ν in example 2.1, the identity functor $Id : Ch_*(\mathcal{A}) \rightarrow Ch_*(\mathcal{A})$ is clearly a descent functor between $(Ch_*(\mathcal{A}), S, s, \mu, \lambda)$ and $(Ch_*(\mathcal{A}), S, s, \nu, \lambda)$. The same can be said using W as equivalences. On the other hand, also $Id : (Ch_*(\mathcal{A}), S, s, \mu, \lambda) \rightarrow (Ch_*(\mathcal{A}), W, s, \mu, \lambda)$ is a descent functor as well.

2.- In the abelian case we can also consider the *normalized simple* $s_N : \Delta^\circ Ch_*(\mathcal{A}) \rightarrow Ch_*(\mathcal{A})$. It is defined as $s_N X = sX/DX$, where DX is the degenerate part of the simplicial chain complex X [May]. The Alexander-Whitney and Eilenberg-Zilber maps preserve degenerate parts. Then, they induce natural transformations μ_N and ν_N between $s_N s_N$ and $s_N D$. In this way we obtain the simplicial descent category $(Ch_*(\mathcal{A}), W, s_N, \mu_N, \lambda_N)$. Note that λ_N is the identity natural transformation in this case, and again we can use ν_N instead of μ_N .

The normalization theorem [May, 22.3] implies that s and s_N are naturally homotopic, so the identity functor induces descent functors between $(Ch_*(\mathcal{A}), W, s, \mu, \lambda)$ and $(Ch_*(\mathcal{A}), W, s_N, \mu_N, \lambda_N)$. The same holds if we replace W by the homotopy equivalences, or μ, μ_N by ν, ν_N .

3.- If $F : I \rightarrow J$ is a functor of small categories and \mathcal{D} is a simplicial descent category, then $F^* : J\mathcal{D} \rightarrow I\mathcal{D}$ is a descent functor with respect to the object-wise descent structures given in proposition 2.6. Also, if $i \in I$, then functor $ev_i : I\mathcal{D} \rightarrow \mathcal{D}$ with $ev_i(f) = f(i)$ is a descent functor. In addition, if $\psi : \mathcal{D} \rightarrow \mathcal{D}'$ is a descent functor, so is the induced functor $\psi : I\mathcal{D} \rightarrow I\mathcal{D}'$.

We state now the *transfer lemma*, that will be widely used in next section.

PROPOSITION 3.3. Consider the data $(\mathcal{D}, s, \mu, \lambda)$ satisfying

(S1) \mathcal{D} is a category with finite coproducts, initial object 0 and final object $*$.

(S3)' $s : \Delta^\circ \mathcal{D} \rightarrow \mathcal{D}$ is a functor.

(S4)' $\mu : sD \rightarrow ss$ is a natural transformation.

(S5)' $\lambda : s(- \times \Delta) \rightarrow Id_{\mathcal{D}}$ is a natural transformation such that there exists $\rho : Id_{\mathcal{D}} \rightarrow s(- \times \Delta)$ with $\lambda\rho = Id$. In addition, λ is compatible with μ , in the sense that the compositions (3) are the identity in \mathcal{D} .

Suppose that $(\mathcal{D}', E', s', \mu', \lambda')$ is a simplicial descent category and $\psi : \mathcal{D} \rightarrow \mathcal{D}'$ a functor such that (DF1) and (DF2) hold. Assume moreover that

(DF3) Θ is represented by a zig-zag of functors and natural transformations, such that all natural transformations are (object-wise) equivalences.

Then $(\mathcal{D}, E, s, \mu, \lambda)$ is a simplicial descent category, where $E = \{f \mid \psi(f) \in E'\}$. In addition, $\psi : \mathcal{D} \rightarrow \mathcal{D}'$ is a descent functor.

REMARK 3.4.

1.- The assumption (DF3) is indeed redundant. By lemma 2.6, $(\mathcal{C} = \text{Fun}(\Delta^\circ \mathcal{D}, \mathcal{D}'), E')$ is a simplicial descent category. But in [R1, proposition 2.6] it is proven that any morphism of $\mathcal{C}[E'^{-1}]$ is represented by a roof $\rightarrow \tilde{\leftarrow}$. Therefore, as E' is saturated, each isomorphism Θ of $\mathcal{C}[E'^{-1}]$ is represented by a roof formed by

two equivalences.

2.- In **(S 4)'** we can assume also that $\mu : \mathbf{ss} \rightarrow \mathbf{sD}$. In this case the compatibility with λ means that $\lambda_{\mathbf{s}X} = \mu_{\Delta \times X}$ and $\mathbf{s}\lambda_X = \mu_{X \times \Delta}$ in \mathcal{D} , for all $X \in \Delta^\circ \mathcal{D}$.

Proof. Assume that $\psi \mathbf{s} \xleftarrow{\Theta^0} A \xrightarrow{\Theta^1} \mathbf{s}'\psi$ is a functorial zig-zag in \mathcal{D}' defining Θ (the general case is completely similar). Let us check that $(\mathcal{D}, \mathbf{E}, \mathbf{s}, \mu, \lambda)$ is a simplicial descent category.

(S 2) \mathbf{E} is a saturated class by definition, and it is stable under coproducts by **(DF 1)**.

(S 3) Given any functor $F : \mathcal{A} \rightarrow \mathcal{D}$, denote by $\sigma_F : F(A) \sqcup F(B) \rightarrow F(A \sqcup B)$ the canonical map. If $X, Y \in \Delta^\circ \mathcal{D}$, we must see that $\psi(\sigma_{\mathbf{s}}) : \psi(\mathbf{s}(X) \sqcup \mathbf{s}(Y)) \rightarrow \psi(\mathbf{s}(X \sqcup Y))$ is in \mathbf{E}' . Since \mathbf{s}' preserves degree-wise equivalences, then $\mathbf{s}'(\sigma_{\psi}) : \mathbf{s}'(\psi X \sqcup \psi Y) \rightarrow \mathbf{s}'\psi(X \sqcup Y)$ is in \mathbf{E}' . As $\sigma_{\mathbf{s}'\psi} : (\mathbf{s}'\psi X) \sqcup (\mathbf{s}'\psi Y) \rightarrow \mathbf{s}'\psi(X \sqcup Y)$ is the composition of $\mathbf{s}'(\sigma_{\psi})$ with $\sigma_{\mathbf{s}'} : (\mathbf{s}'\psi X) \sqcup (\mathbf{s}'\psi Y) \rightarrow \mathbf{s}'(\psi X \sqcup \psi Y)$, then $\sigma_{\mathbf{s}'\psi} \in \mathbf{E}'$. Hence **(S 3)** follows from the commutative diagram

$$\begin{array}{ccccccc} \psi(\mathbf{s}X \sqcup \mathbf{s}Y) & \xleftarrow{\sigma_{\psi}} & \psi \mathbf{s}X \sqcup \psi \mathbf{s}Y & \xleftarrow{\Theta_X^0 \sqcup \Theta_Y^0} & A(X) \sqcup A(Y) & \xrightarrow{\Theta_X^1 \sqcup \Theta_Y^1} & \mathbf{s}'(\psi X) \sqcup \mathbf{s}'(\psi Y) \\ \downarrow \psi(\sigma_{\mathbf{s}}) & & \downarrow \sigma_{\psi \mathbf{s}} & & \downarrow \sigma_A & & \downarrow \sigma_{\mathbf{s}'\psi} \\ \psi \mathbf{s}(X \sqcup Y) & \xleftarrow{Id} & \psi \mathbf{s}(X \sqcup Y) & \xleftarrow{\Theta_{X \sqcup Y}^0} & A(X \sqcup Y) & \xrightarrow{\Theta_{X \sqcup Y}^1} & \mathbf{s}'\psi(X \sqcup Y) \end{array}$$

On the other hand, **(S 4)** and **(S 5)** are direct consequences of the compatibility between $\Theta, \lambda, \lambda'$ and Θ, μ, μ' . Axiom **(S 6)** follows from **(S 6)** for \mathbf{s}' and from the existence of Θ , which implies also **(S 8)**. Therefore, it remains to see **(S 7)**. Given a morphism $f : X \rightarrow Y$ in \mathcal{D} we have to prove that $\psi f \in \mathbf{E}'$ if and only if $\psi \mathbf{s}(Cf) \rightarrow \psi(*)$ is so. By **(DF 2)**, the last condition is equivalent to $\mathbf{s}'\psi(Cf) \rightarrow \psi(*) \in \mathbf{E}'$.

Recall that $\psi(Cf)_n = \psi(Y_n \sqcup X_n \sqcup \dots \sqcup *)$. By **(DF 1)**, $\psi(Cf)_n$ is equivalent to $Cyl(\psi(f), \epsilon)_n = (\psi Y_n) \sqcup (\psi X_n) \sqcup \dots \sqcup (\psi *)$, where $\epsilon : \psi(X) \rightarrow \psi(*)$ is equal to $\psi(X \rightarrow *)$. By **(S 3)** and **(S 6)**, it is enough to prove that $\psi f \in \mathbf{E}'$ if and only if $\mathbf{s}'Cyl(\psi f, \epsilon) \rightarrow \psi(*)$ is so. But $\mathbf{s}'Cyl(\psi f, \epsilon) \rightarrow \psi(*)$ is in \mathbf{E}' if and only if the simple of the canonical map $I : \psi(*) \rightarrow Cyl(\psi(f), \epsilon)$ is in \mathbf{E}' , by the 2-out-of-3 property. Hence, **(S 7)** follows from proposition 2.9. \square

4 Simplicial Examples

Our first example of simplicial descent category is the one of chain complexes 2.5. Using the transfer lemma, we deduce new simplicial descent structures, following the picture below

$$\begin{array}{ccccccc} & \text{Singular} & & \text{Free} & & & \\ & \text{Chain} & & \text{Module} & & & \\ Top & \xrightarrow{\quad} & \Delta^\circ Set & \xrightarrow{\quad} & \Delta^\circ \mathcal{A} & \xrightarrow{K} & Ch \mathcal{A} \end{array}$$

Simplicial objects in additive/abelian categories

Let $Ch_+ \mathcal{A}$ be the category of positive chain complexes over an additive or abelian category \mathcal{A} . As in 2.1, the functor $K : \Delta^\circ \mathcal{A} \rightarrow Ch_+ \mathcal{A}$ is defined by taking the alternate sum of the face maps of X as boundary map of KX . The Eilenberg-Zilber-Cartier theorem [DP] means in the simplicial descent context that functor K transfers the descent structure from $Ch_+ \mathcal{A}$ to $\Delta^\circ \mathcal{A}$. Define a descent structure on $\Delta^\circ \mathcal{A}$ by:

Simple functor: The simple functor is the diagonal functor $D : \Delta^\circ \Delta^\circ \mathcal{A} \rightarrow \Delta^\circ \mathcal{A}$.

Equivalences: We consider two classes of equivalences. On one hand, $S = \{\text{simplicial homotopy equivalences}\}$. On the other, $W = \{\text{quasi-isomorphisms (that is, those maps inducing isomorphism on homology)}\}$.

Transformations λ and μ : The natural transformations λ and μ are defined as the corresponding identity natural transformations.

PROPOSITION 4.1. ${}_H\Delta^\circ\mathcal{A} = (\Delta^\circ\mathcal{A}, S, D, \mu, \lambda)$ and ${}_Q\Delta^\circ\mathcal{A} = (\Delta^\circ\mathcal{A}, W, D, \mu, \lambda)$ are additive simplicial descent categories. In addition, $K : {}_*\Delta^\circ\mathcal{A} \rightarrow {}_*Ch_+\mathcal{A}$ is a descent functor for $*$ = H, Q , where ${}_HCh_*(\mathcal{A})$ and ${}_QCh_*(\mathcal{A})$ are the descent structures given in example 2.5.

Proof. It follows from the Eilenberg-Zilber-Cartier theorem [DP] that $K : {}_*\Delta^\circ\mathcal{A} \rightarrow {}_*Ch_+\mathcal{A}$ satisfies the transfer lemma 3.3 for $*$ = H, Q , taking Θ = Alexander-Whitney map, which is a natural homotopy equivalence $KD \rightarrow sK$. On the other hand, the equality $\{f \mid Kf \text{ is a homotopy equivalence}\} = S$ is deduced from the classical homotopy theory of simplicial objects in \mathcal{A} [May]. \square

Simplicial Sets

Let Set be the category of sets, mod_R the one of modules over the ring R , and $L : Set \rightarrow mod_R$ the functor that maps T to the free R -module with basis T . Recall that the homology of a simplicial set W (with coefficients in R) is the homology of the simplicial R -module $L(W)$. That is, it is the homology of the chain complex $KL(W)$. Define a descent structure on $\Delta^\circ\mathcal{A}$ by:

Simple functor: The simple functor is the diagonal $D : \Delta^\circ\Delta^\circ Set \rightarrow \Delta^\circ Set$.

Equivalences: The class E of equivalences consists of those morphisms that induce isomorphism on homology (with coefficients in R).

Transformations λ and μ : The natural transformations λ and μ are defined as the identity natural transformations.

The statement of the transfer lemma for $L : \Delta^\circ Set \rightarrow {}_Q\Delta^\circ mod_R$ yields

PROPOSITION 4.2. $(\Delta^\circ Set, E, D, \mu, \lambda)$ is a simplicial descent category, and $L : \Delta^\circ Set \rightarrow {}_Q\Delta^\circ mod_R$ is a descent functor.

Topological Spaces

Let $\Delta^m \subset \mathbb{R}^{m+1}$ be the standard m -dimensional simplex. Define the following simplicial descent structure on the category Top of topological spaces and continuous maps.

Simple functor: $s : \Delta^\circ Top \rightarrow Top$ is the ‘fat’ geometric realization [S, appendix A]. A simplicial topological space X induces the bifunctor $\Delta_e \times \Delta_e \rightarrow Top$, $([n], [m]) \rightarrow X_n \times \Delta^m$. The fat geometric realization of X is the coend of this bifunctor [ML]:

$$sX = \int^n X_n \times \Delta^n$$

More specifically, sX is the quotient space of $\coprod_{n \geq 0} X_n \times \Delta^n$ modulo the equivalence relation generated by $(d_i(x), u) \sim (x, d^i(u))$ if $(x, u) \in X_n \times \Delta^{n-1}$. We will write $[x, t]$ for the equivalence class of an element $(x, t) \in \coprod_{n \geq 0} X_n \times \Delta^n$.

Equivalences: The class E consists of those maps that induce isomorphism on singular homology with coefficients in a fixed commutative ring R .

Transformation λ : Given $X \in Top$, define $\lambda_X : s(X \times \Delta) \rightarrow X$ as $\lambda_X[x, t] = x$.

Transformation μ : If $Z \in \Delta^\circ\Delta^\circ Top$, define $\mu_Z : sDZ \rightarrow ssZ$ as $\mu_Z([z_{nn}, t_n]) = [[z_{n,n}, t_n], t_n]$.

PROPOSITION 4.3. $(Top, E, s, \mu, \lambda)$ is a simplicial descent category. In addition, the singular functor $S : Top \rightarrow \Delta^\circ Set$ is a descent functor.

The proof is basically [Dup, 5.15]. See [R, 5.5] for more details.

REMARK 4.4. I.- Fat geometric realization is exact with respect to quasi-isomorphisms [Dup, 5.16]. If we consider the usual geometric realization $|\cdot| : \Delta^\circ \text{Top} \rightarrow \text{Top}$, it turns out that given a map $f : X \rightarrow Y$ such that f_n is a quasi-isomorphism for all n , we need to impose some extra conditions in order to have that $|f|$ is again a quasi-isomorphism (see, for instance, [M, 11.13]). This is the reason why we do not consider $|\cdot|$ as simple functor.

On the other hand, recall that the canonical map $\mathbf{s}X \rightarrow |X|$ is a homotopy equivalence in case the simplicial topological space X is such that its degeneracy maps are closed cofibrations (see [S, appendix A]).

II.- It can be proved as well that geometric realization $|\cdot| : \Delta^\circ \text{Set} \rightarrow \text{Top}$ and fat geometric realization $\mathbf{s} : \Delta^\circ \text{Set} \rightarrow \text{Top}$ are descent functors [R, 5.5].

5 Cosimplicial Examples

Recall that a cosimplicial descent category is by definition the dual category of a simplicial descent category. Hence, cochain complexes form a cosimplicial descent category. We describe this structure in our first cosimplicial example. The remaining examples are deduced from the dual of transfer lemma 3.3.

Cochain complexes

Let $Ch^*(\mathcal{A})$ be the category of (unbounded) cochain complexes over an additive or abelian category \mathcal{A} with countable products.

From equality $(Ch_*(\mathcal{A}^{op}))^{op} = Ch^*(\mathcal{A})$ we get the data $(Ch^*\mathcal{A}, \mathbf{s}, \mu, \lambda)$. Together with the homotopy equivalences or quasi-isomorphisms as equivalences, this is in fact a cosimplicial descent category by definition.

More concretely, if $X \in \Delta Ch^*(\mathcal{A})$ then $(\mathbf{s}X)^q = \prod_{p+n=q} X^{n,p}$, and the boundary map is $d = \prod (-1)^p \partial + d^{X^n} : \prod_{p+n=q} X^{n,p} \longrightarrow \prod_{p+n=q+1} X^{n,p}$.

On the other hand, a cochain complex A is canonically a direct summand of $\mathbf{s}(A \times \Delta)$, and $\lambda_A : A \rightarrow \mathbf{s}(A \times \Delta)$ is the inclusion. Finally, $\mu : \mathbf{s}\mathbf{s} \rightarrow \mathbf{s}\mathbf{D}$ and $\nu : \mathbf{s}\mathbf{D} \rightarrow \mathbf{s}\mathbf{s}$ are the (dual) Alexander-Whitney and Eilenberg-Zilber maps, respectively.

(5.1) As in the case of chain complexes, we can define the *normalized* simple $\mathbf{s}_N : \Delta Ch^*(\mathcal{A}) \rightarrow Ch^*(\mathcal{A})$. It is given by $(\mathbf{s}_N A)^n = \prod_{p+q=n} \bigcap_{i=0}^{p-1} \text{Ker}\{s_i : A^{p,q} \rightarrow A^{p-1,q}\}$. Now λ_N is the identity natural transformation, and μ_N, ν_N are induced by the previous ones.

Differential graded algebras

Consider a commutative (associative and unitary) ring R . Let Ch^+R be the category of positive cochain complexes of R -modules and $\mathbf{Dga}(R)$ the one of differential graded R -algebras (not necessarily commutative, and positively graded).

Recall that $\mathbf{s} : \Delta Ch^+R \rightarrow Ch^+R$ is the composition of functors K and Tot , which are monoidal with respect to the tensor product of R -modules. Then so is \mathbf{s} . Given cosimplicial cochain complexes X and Y , the Künneth morphism $k : \mathbf{s}X \otimes \mathbf{s}Y \longrightarrow \mathbf{s}(X \otimes Y)$ is induced by the Alexander-Whitney map. In degree n , k^n is the sum of the morphisms

$$(-1)^{i(t+s)} X(d^0 \overset{s}{\dots} d^0) \otimes Y(d^{i+s} d^{i+s-1} \dots d^{s+1}) : X^{i,j} \otimes Y^{s,t} \longrightarrow X^{i+s,j} \otimes Y^{i+s,t}$$

This Künneth morphism preserves degenerate parts, so it gives rise to a Künneth morphism for the normalized simple, $k : \mathbf{s}_N X \otimes \mathbf{s}_N Y \longrightarrow \mathbf{s}_N(X \otimes Y)$.

Hence \mathbf{s}_N induces $\mathbf{s}_{AW} : \Delta \mathbf{Dga}(R) \rightarrow \mathbf{Dga}(R)$, the (normalized) Alexander-Whitney simple [N, 3.1], which is part of the following descent structure.

Simple functor: Given $A \in \Delta \mathbf{Dga}(R)$, consider the cosimplicial cochain complex $UA \in \Delta Ch^* R$ obtained by forgetting the multiplicative structure of A . Then $\mathbf{s}_N(UA)$ is a differential graded algebra through $\mathbf{s}_N(UA) \otimes \mathbf{s}_N(UA) \rightarrow \mathbf{s}_N(UA)$, defined as the composition

$$\mathbf{s}_N(UA) \otimes \mathbf{s}_N(UA) \xrightarrow{k} \mathbf{s}_N(UA \otimes UA) \xrightarrow{\mathbf{s}_N \tau} \mathbf{s}_N(A)$$

where k is the Künneth morphism and $\tau^n : UA^n \otimes UA^n \rightarrow UA^n$ is the structural morphism of the differential graded algebra A^n . The Alexander-Whitney simple is the functor $\mathbf{s}_{AW} : \Delta \mathbf{Dga}(R) \rightarrow \mathbf{Dga}(R)$ obtained in this way.

Transformations λ_{AW} and μ_{AW} : The transformation λ_{AW} is equal to the identity natural transformation. On the other hand, $\mu_{AW} : \mathbf{s}_{AW} D \rightarrow \mathbf{s}_{AW} \mathbf{s}_{AW}$ is obtained from the Eilenberg-Zilber map ν_N at the level of cochain complexes. In fact, it is a morphism of algebras [EM, p. 232].

REMARK 5.2. We use the Eilenberg-Zilber map ν_N as the datum μ_{AW} in the cosimplicial descent structure of $\mathbf{Dga}(R)$ because it is symmetric, so it preserves the algebra structure. This is not true for the Alexander-Whitney map, which is just symmetric up to homotopy.

The following proposition follows directly from the transfer lemma.

PROPOSITION 5.3. *The category $\mathbf{Dga}(R)$ together with the normalized Alexander-Whitney simple and the quasi-isomorphisms as equivalences is a cosimplicial descent category. In addition, the forgetful functor from $\mathbf{Dga}(R)$ to ${}_{\mathcal{Q}}Ch^* R$ is a descent functor.*

Commutative differential graded algebras

Denote by $\mathbf{Cdga}(k)$ the category of commutative differential graded algebras (or cdg algebras) over a fixed field k of characteristic 0. Navarro's Thom-Whitney simple [N] gives rise to a cosimplicial descent structure on $\mathbf{Cdga}(k)$, which we now describe.

Simple functor: Let $L \in \Delta^\circ \mathbf{Cdga}(k)$ [BG] be

$$L_n = \frac{\Lambda(x_0, \dots, x_n, dx_0, \dots, dx_n)}{(\sum x_i - 1, \sum dx_i)},$$

where $\Lambda(x_0, \dots, x_n, dx_0, \dots, dx_n)$ is the free cdg algebra generated by $\{x_k\}_k$ in degree 0 and by $\{dx_k\}_k$ in degree 1. The boundary map is the unique derivation such that $d(x_k) = dx_k$, $d(dx_k) = 0$.

The face maps $d_i : L_{n+1} \rightarrow L_n$ and the degeneracy maps $s_j : L_n \rightarrow L_{n+1}$ are defined as $d_i(x_k) =$

$$\begin{cases} x_k, & k < i \\ 0, & k = i \\ x_{k-1}, & k > i \end{cases} \text{ and } s_j(x_k) = \begin{cases} x_k, & k < j \\ x_k + x_{k+1}, & k = j \\ x_{k+1}, & k > j \end{cases}.$$

Given $A \in \Delta \mathbf{Cdga}(k)$, consider the bifunctor $L \otimes A : \Delta_e^\circ \times \Delta_e \rightarrow \mathbf{Cdga}(k)$, $([n], [m]) \rightarrow L_n \otimes A^m$. Recall that Δ_e is the strict simplicial category (with no degeneracy maps). Then, the Thom-Whitney simple is the end [ML]

$$\mathbf{s}_{TW}(A) = \int_n L_n \otimes A^n.$$

Equivalences: The class E consists of the quasi-isomorphisms, that is to say, those morphisms of cdg algebras which induce isomorphism on cohomology.

Transformation λ : If $A \in \mathbf{Cdga}(k)$, the morphisms $A \rightarrow A \otimes L_n$; $a \rightarrow a \otimes 1$ give rise to $\lambda_{TW} : A \rightarrow s_{TW}(A \times \Delta)$.

Transformation μ : If $Z \in \Delta\Delta\mathbf{Cdga}(k)$, $\mu_{TWZ} : s_{TW}s_{TW}Z \rightarrow s_{TW}DZ = \int_p Z^{p,p} \otimes L_p$ is given by the morphisms

$$s_{TW}s_{TW}Z \xrightarrow{\pi} Z^{p,p} \otimes L_p \otimes L_p \xrightarrow{Id \otimes \tau_p} Z^{p,p} \otimes L_p$$

where π is the iterated projection and $\tau_p : L_p \otimes L_p \rightarrow L_p$ is the structural map of the cdg algebra L_p , which is a morphism of cdg algebras since L_p is commutative.

Let $k[p]$ be the cochain complex with $(k[p])^p = k$ and $(k[p])^n = 0$ if $n \neq p$. As in [N, (2.2)], we denote by $\int_{\Delta^p} : L_p \rightarrow k[p]$ the map of cochain complexes which in degree p is the *integral over the simplex Δ^p* , $L_p^p \rightarrow k$.

PROPOSITION 5.4. *The category $\mathbf{Cdga}(k)$ together with the Thom-Whitney simple and the quasi-isomorphisms is a cosimplicial descent category. In addition, the forgetful functor from $\mathbf{Cdga}(k)$ to the descent category QCh^*k of cochain complexes of k -vector spaces is a descent functor.*

Proof. If $A \in \mathbf{Cdga}(k)$, define $\rho_A : s_{TW}(A \times \Delta) \rightarrow A$ as the projection $\int_n L_n \otimes A \rightarrow L_0 \otimes A \equiv k \otimes A \equiv A$. To see that the forgetful functor $U : \mathbf{Cdga}(k) \rightarrow Ch^*k$ satisfies the dual hypothesis of proposition 3.3, let us check $(FD2)^{op}$. By [N, 2.15], the diagram

$$\begin{array}{ccc} \Delta\mathbf{Cdga}(k) & \xrightarrow{U} & \Delta Ch^*k \\ s_{TW} \downarrow & & \downarrow s \\ \mathbf{Cdga}(k) & \xrightarrow{U} & Ch^*k. \end{array}$$

commutes up to a natural homotopy equivalence $I : Us_{TW} \rightarrow sU$. Given $A \in \Delta\mathbf{Cdga}(k)$, $I_A : Us_{TW}A \rightarrow sUA$ is in degree n the map $(s_{TW}A)^n \rightarrow (sUA)^n = \prod_{p+q=n} A^{p,q}$ whose projection onto the component $A^{p,q}$ is given by

$$(s_{TW}(A))^n = \int_m (A^m \otimes L_m)^n \xrightarrow{\pi} (A^p \otimes L_p)^n \xrightarrow{Id \otimes \int_{\Delta^p}} A^{p,q},$$

where π denotes the projection. The compatibility between $\lambda_{TW} : Id_{\mathbf{Cdga}(k)} \rightarrow s_{TW}(- \times \Delta)$, $\lambda : Id_{Ch^*k} \rightarrow s(- \times \Delta)$ and $I = \Theta$ follows from the equality $I_{A \times \Delta} U(\lambda_{TW} A) = \lambda_{UA}$. It remains to see that $\mu_{TW} : s_{TW}s_{TW} \rightarrow s_{TW}D$, $\mu : ss \rightarrow sD$ and Θ are compatible. If $Z \in \Delta\Delta\mathbf{Cdga}(k)$, I claim that $\Theta_{DZ}U(\mu_{TW})_Z$ is equal, up to homotopy, to $\mu_{UZ}s\Theta_Z\Theta_{s_{TW}Z}$. Recall that the transformation μ comes from the Alexander-Whitney map.

Using the homotopy inverse E of Θ given in [N], one can obtain a homotopy inverse E' of $s\Theta_Z\Theta_{s_{TW}Z}$. Then, it is enough to see that $\Theta_{DZ}U(\mu_{TW})_ZE'$ coincides with μ_{UZ} up to homotopy.

As in [N, (3.4)], one can check that $\Theta_{DZ}U(\mu_{TW})_ZE'$ is the total complex of a map of double complexes which is equal to the Alexander-Whitney map in degree 0, so the statement follows from Eilenberg-Zilber-Cartier theorem [DP]. \square

DG-modules over a DG-category

Let R be a fixed commutative ring. We follow the notations given in [K] for DG-categories. Recall that $\text{Dif } R$ is the DG-category whose objects are (unbounded) cochain complexes of R -modules, and \mathcal{CA} is the category of *differential graded modules* (or DG-modules) over a fixed DG-category \mathcal{A} . Recall that a DG-module M is a functor of DG-categories $M : \mathcal{A}^{op} \rightarrow \text{Dif } R$. A morphism of DG-modules is a natural transformation $\lambda : M \rightarrow N$ such that $\lambda_A : MA \rightarrow NA$ is a morphism of cochain complexes for all $A \in \mathcal{A}$.

Next we endow \mathcal{CA} with a cosimplicial descent category structure. To this end, denote by ${}_QCh^*R = (Ch^*R, {}_RW, {}_R\mathbf{s}, {}_R\mu, {}_R\lambda)$ the descent category of cochain complexes of R -modules.

Simple functor: Given $M = \{M^n, d^i, s^j\} \in \Delta\mathcal{CA}$ and $A \in \mathcal{A}$, note that $MA = \{M^n A, d_A^i, s_A^j\}$ is a cosimplicial cochain complex of R -modules. The image under $\mathbf{s}M : \mathcal{A}^\circ \rightarrow \text{Dif } R$ of A is the cochain complex

$$(\mathbf{s}M)A := {}_R\mathbf{s}(MA) \text{ which is in degree } m \quad ((\mathbf{s}M)A)^m = \prod_{p+q=m} (M^p A)^q.$$

If $f : A \rightarrow B$ is a map of degree r in \mathcal{A} , then $M^n f : M^n B \rightarrow M^n A$ is $M^n f = \{M^n f^k : (M^n B)^k \rightarrow (M^n A)^{k+r}\}_{k \in \mathbb{Z}}$, and $(\mathbf{s}M)f$ is defined as

$$(\mathbf{s}M)f = \{((\mathbf{s}M)f)^k\}_{k \in \mathbb{Z}} \text{ where } ((\mathbf{s}M)f)^k = \prod_{p+q=k+r} M^p f^{q-r} : ((\mathbf{s}M)B)^k \rightarrow ((\mathbf{s}M)A)^{k+r}$$

If $\tau : M \rightarrow N$ is a morphism of $\Delta\mathcal{CA}$ then $(\mathbf{s}\tau)_A = {}_R\mathbf{s}(\tau_A) : (\mathbf{s}M)A \rightarrow (\mathbf{s}N)A$.

Equivalences: The equivalences are those morphisms $\rho : M \rightarrow N$ such that $\rho_A : MA \rightarrow NA$ is a quasi-isomorphism in Ch^*R , for all A in \mathcal{A} .

Transformations λ and μ : Given $M \in \mathcal{CA}$ and $Z \in \Delta\mathcal{CA}$, transformations $\lambda(M) : M \rightarrow \mathbf{s}(M \times \Delta)$ and $\mu(Z) : \mathbf{ss}Z \rightarrow \mathbf{s}DZ$ are respectively

$$\lambda(M)_A = {}_R\lambda_{MA} \text{ and } \mu(Z)_A = {}_R\mu_{ZA} \text{ for each object } A \text{ of } \mathcal{A}.$$

Next proposition follows from the fact that ${}_QCh^*R$ is a cosimplicial descent category.

PROPOSITION 5.5. *$(\mathcal{CA}, \mathbf{s}, E, \lambda, \mu)$ is an additive cosimplicial descent category.*

Filtered cochain complexes

Given an abelian category \mathcal{A} , let $\text{CF}^+\mathcal{A}$ be the category of filtered positive cochain complexes. Its objects are pairs (A, F) , where A is a positive cochain complex over \mathcal{A} and F is a decreasing birregular filtration. We endow here $\text{CF}^+\mathcal{A}$ with two different descent structures. In the first one, the equivalences are the filtered quasi-isomorphisms, and they are the E_2 -isomorphism in the second one. Both are related through the ‘decalage’ functor [DeII, I.3.3].

We briefly recall the *graded* and *decalage* functors. Let $(Ch^+\mathcal{A}, \mathbf{s}, W, \mu, \lambda)$ be the descent category of positive cochain complexes over \mathcal{A} .

For each $k \in \mathbb{Z}$, the *graded* functor $\mathbf{Gr}_k : \text{CF}^+\mathcal{A} \rightarrow Ch^+\mathcal{A}$ is given by ${}_F\mathbf{Gr}_k A = F^k A / F^{k+1} A$, where $(A, F) \in \text{CF}^+\mathcal{A}$. Let $(Ch^+\mathcal{A})^{\mathbb{Z}}$ be the category of graded cochain complexes, whose objects are families, indexed over \mathbb{Z} , of positive cochain complexes. The *graded* functor $\mathbf{Gr} : \text{CF}^+\mathcal{A} \rightarrow (Ch^+\mathcal{A})^{\mathbb{Z}}$ applied to (A, F) is in degree k the complex ${}_F\mathbf{Gr}_k A$.

Let $\text{Dec} : \text{CF}^+\mathcal{A} \rightarrow \text{CF}^+\mathcal{A}$ be the functor that maps (A, F) to the filtered complex $(A, \text{Dec}(F))$, where

$Dec(F)$ is the ‘decalage’ filtration of F

$$Dec(F)^p A^n = Z_1^{p+n, -p} = \ker \left\{ d : F^{p+n} A^n \rightarrow \frac{A^{n+1}}{F^{p+n+1} A^{n+1}} \right\}.$$

Next we describe the first descent structure on $CF^+ \mathcal{A}$.

Simple functor: If $(A, F) \in \Delta CF^+ \mathcal{A}$, let $\mathbf{s}F$ be the filtration of $\mathbf{s}A$ defined as $(\mathbf{s}F)^k(\mathbf{s}A) = \mathbf{s}(F^k A)$. Define $(\mathbf{s}, \mathbf{s}) : \Delta CF^+ \mathcal{A} \rightarrow CF^+ \mathcal{A}$ as $(\mathbf{s}, \mathbf{s})(A, F) = (\mathbf{s}A, \mathbf{s}F)$.

Equivalences: The class E consists of the filtered quasi-isomorphisms, that is, those f such that $\mathbf{Gr}_k(f)$ is a quasi-isomorphism for all $k \in \mathbb{Z}$.

Transformations λ and μ : Transformations λ and μ in $Ch^+ \mathcal{A}$ preserve the filtrations, inducing λ and μ for (\mathbf{s}, \mathbf{s}) .

PROPOSITION 5.6. $(CF^+ \mathcal{A}, (\mathbf{s}, \mathbf{s}), E, \lambda, \mu)$ is an additive cosimplicial descent category. In addition, $\mathbf{Gr} : CF^+ \mathcal{A} \rightarrow (Ch^+ \mathcal{A})^{\mathbb{Z}}$ is a descent functor.

Proof. Note that $(Ch^+ \mathcal{A})^{\mathbb{Z}}$ is a cosimplicial descent category by 2.6. The transfer lemma is trivially satisfied for the graded functor $\mathbf{Gr} : CF^+ \mathcal{A} \rightarrow (Ch^+ \mathcal{A})^{\mathbb{Z}}$. Indeed, $\mathbf{sGr}(A, F) \equiv \mathbf{Gr}(\mathbf{s}A, \mathbf{s}F)$ since \mathbf{s} is exact. \square

Now we endow $CF^+ \mathcal{A}$ with a second descent structure.

Simple functor: Define $(\mathbf{s}, \delta) : \Delta CF^+ \mathcal{A} \rightarrow CF^+ \mathcal{A}$ as follows. If $(A, F) \in CF^+ \mathcal{A}$, then $(\mathbf{s}, \delta)(A, F) = (\mathbf{s}A, \delta F)$, where $\mathbf{s}A$ is the usual simple of cochain complexes. On the other hand, δF is the *diagonal filtration* over $\mathbf{s}A$, given by

$$(\delta F)^k(\mathbf{s}A)^n = \bigoplus_{i+j=n} F^{k-i} A^{i,j}.$$

Equivalences: The class of equivalences E_2 consists of the E_2 -isomorphisms, that is, those morphisms that induce isomorphism on the second term of the spectral sequences associated with (A, F) and (B, G) .

Transformations λ and μ : Transformations λ and μ in $Ch^+ \mathcal{A}$ preserve the filtrations, inducing λ and μ for (\mathbf{s}, δ) .

PROPOSITION 5.7. $(CF^+ \mathcal{A}, (\mathbf{s}, \delta), E_2, \lambda, \mu)$ is an additive cosimplicial descent category. In addition $Dec : CF^+ \mathcal{A} \rightarrow CF^+ \mathcal{A}$ is a descent functor from $(CF^+ \mathcal{A}, (\mathbf{s}, \delta), E_2, \lambda, \mu)$ to $(CF^+ \mathcal{A}, (\mathbf{s}, \mathbf{s}), E, \lambda, \mu)$.

Proof. First of all, it follows from [DeII, I.3.4] that $E_2 = \{f \in CF^+ \mathcal{A} \mid Dec(f) \in E\}$. Property **(DF 1)** holds trivially for Dec , since it is an additive functor. On the other hand, **(DF 2)** follows from the equality $(\mathbf{s}, \mathbf{s})Dec = Dec(\mathbf{s}, \delta)$ [DeIII, 8.I.16]. \square

REMARK 5.8. I. If k is a fixed integer, then previous proposition can be proved in a similar way for the category $CF^k \mathcal{A}$ of filtered (by a birregular filtration) cochain complexes (A, F) such that $A^n = 0$ when $n < k$. **II.** The above simplicial descent structures can be adapted as well to the normalized case, replacing \mathbf{s} by the normalized simple \mathbf{s}_N .

Mixed Hodge Complexes

We define here a category of mixed Hodge complexes and endow it with a structure of cosimplicial descent category. The simple functor agrees with Deligne’s construction [DeIII, 8.I.15] on objects. It becomes a functor since the comparison morphisms of our mixed Hodge complexes are genuine filtered quasi-isomorphisms instead of maps in the corresponding filtered derived category.

We denote by \mathbb{Q} and \mathbb{C} the respective categories of \mathbb{Q} and \mathbb{C} -vector spaces.

DEFINITION 5.9. A *mixed Hodge complex* is the data $((K_{\mathbb{Q}}, W), (K_{\mathbb{C}}, W, F), \alpha)$, where

- a) $(K_{\mathbb{Q}}, W)$ is a positive cochain complex of \mathbb{Q} -vector spaces with finite dimensional cohomology, filtered by the increasing filtration W .
- b) $(K_{\mathbb{C}}, W, F)$ is a positive bifiltered cochain complex of \mathbb{C} -vector spaces, where W and F are respectively increasing and decreasing birregular filtrations, called the *weight* and *Hodge* filtrations.
- c) α is the data $(\alpha_0, \alpha_1, (\tilde{K}, \tilde{W}))$, where (\tilde{K}, \tilde{W}) is an object of $\text{CF}^+\mathbb{C}$ and α_i , $i = 0, 1$, is a filtered quasi-isomorphism

$$(K_{\mathbb{C}}, W) \xleftarrow{\alpha_0} (\tilde{K}, \tilde{W}) \xrightarrow{\alpha_1} (K_{\mathbb{Q}}, W) \otimes \mathbb{C} .$$

The following axiom must be satisfied

(MHC) For each n the boundary map of ${}_W\mathbf{Gr}_n K_{\mathbb{C}}$ is compatible with the induced filtration F , and $({}_W\mathbf{Gr}_n H^k K_{\mathbb{C}}, F)$ is a Hodge structure of weight $n + k$. That is,

$${}_F\mathbf{Gr}^p {}_W\overline{F}\mathbf{Gr}^q {}_W\mathbf{Gr}_n H^k K_{\mathbb{C}} = 0 \text{ for } p + q \neq n + k .$$

REMARK 5.10. (1.-) The filtered quasi-isomorphisms have a calculus of fractions in the category of filtered complexes up to filtered homotopy [III, p.271]. Hence any isomorphism in the filtered derived category can be represented by a zig-zag as in c). Moreover, any arbitrary zig-zag of filtered quasi-isomorphisms can be reduced to a length 2 zig-zag in a functorial way, [R, 6.5.17], using a ‘quasi-pullback’ procedure (constructed through the path object coming from the cosimplicial structure on $\text{CF}^+\mathbb{C}$). In [Be] the comparison zig-zag α has the shape $\cdot \leftarrow \cdot \rightarrow \cdot$, because they use a ‘reduction’ procedure based on the filtered cone object, instead of the filtered path one. Another possibility is to allow zig-zags α of arbitrary length, as in [PS]. All constructions and results developed here can be adapted to this situation.

(2.-) Except for the \mathbb{Z} -part, a mixed Hodge complex as above is a mixed Hodge complex in the sense of Deligne, thinking in α as an isomorphism in the filtered derived category. Also, applying the decalage functor Dec to the weight filtration W we get a mixed Hodge complex as defined in [H] and [Be].

(3.-) We dropped the \mathbb{Z} -part of a mixed Hodge complex by simplicity, but all results in this section are also valid for mixed Hodge complexes with \mathbb{Z} -coefficients.

EXAMPLE 5.11. [DeIII, 8.I.8] Let $j : U \rightarrow X$ be an open immersion of complex smooth varieties, where X is proper and $Y = X \setminus U$ is a normal crossing divisor.

If \mathcal{F} is a sheaf on T , set $R\Gamma(T, \mathcal{F}) = \Gamma(T, \mathcal{C}_{\text{God}}\mathcal{F})$, where $\mathcal{C}_{\text{God}}\mathcal{F}$ is the Godement resolution of \mathcal{F} . Analogously, if \mathcal{F} is a bounded below complex of sheaves on T (eventually filtered), set $R\Gamma(T, \mathcal{F}) = \Gamma(T, \text{Tot}(\mathcal{C}_{\text{God}}\mathcal{F}))$, where Tot means the total complex of a double complex. The point is that $R\Gamma$ has values in the category of (filtered) complexes instead of in the derived category, and the (hyper) cohomology $H^*(T, \mathcal{F})$ may be computed as the cohomology of $R\Gamma(T, \mathcal{F})$.

Let $(\Omega_X\langle Y \rangle, W, F)$ be the logarithmic De Rham complex of X along Y [DeII, 3.I]. W is the so-called ‘weight filtration’, and F is the ‘Hodge filtration’, that is the filtration ‘bête’ associated with $\Omega_X\langle Y \rangle$.

Let $(R\Gamma(j_*\mathbb{Q}_U, W)$ be the filtered complex of sheaves of \mathbb{Q} -vector spaces on X , where $W = \tau_{\leq}$ is the ‘canonical’ filtration. A general argument shows that there is a zig-zag of filtered quasi-isomorphisms relating $R\Gamma(j_*\mathbb{Q}_U, W) \otimes \mathbb{C}$ to $R\Gamma(\Omega_X\langle Y \rangle, W)$ (see [H, p. 66] or [PS, 4.11]). It is basically the result [DeII, 3.I.8] relating $\Omega_X\langle Y \rangle$ to $j_*\Omega_U$, together with Poincaré lemma (that is, Ω_U is a resolution of the constant sheaf \mathbb{C}_U). This zig-zag can be reduced to a length 2 zig-zag in a natural way (through the path object). Therefore $(R\Gamma(j_*\mathbb{Q}, W), R\Gamma(\Omega_X\langle Y \rangle, W, F))$ is a mixed Hodge complex in the sense of previous definition.

DEFINITION 5.12. A morphism of mixed Hodge complexes

$$f = (f_{\mathbb{Q}}, f_{\mathbb{C}}, \tilde{f}) : ((K_{\mathbb{Q}}, W), (K_{\mathbb{C}}, W, F), \alpha) \rightarrow ((K'_{\mathbb{Q}}, W'), (K'_{\mathbb{C}}, W', F'), \alpha')$$

consists of morphisms $f_{\mathbb{Q}} : (K_{\mathbb{Q}}, W) \rightarrow (K'_{\mathbb{Q}}, W')$ and $f_{\mathbb{C}} : (K_{\mathbb{C}}, W, F) \rightarrow (K'_{\mathbb{C}}, W', F')$ of (bi)filtered complexes. If $\alpha = (\alpha_0, \alpha_1, \tilde{K}, \tilde{W})$ and $\alpha' = (\alpha'_0, \alpha'_1, \tilde{K}', \tilde{W}')$ are the respective zig-zags

$$(K_{\mathbb{C}}, W) \xleftarrow{\alpha_0} (\tilde{K}, \tilde{W}) \xrightarrow{\alpha_1} (K_{\mathbb{Q}}, W) \otimes \mathbb{C} \quad (K'_{\mathbb{C}}, W') \xleftarrow{\alpha'_0} (\tilde{K}', \tilde{W}') \xrightarrow{\alpha'_1} (K'_{\mathbb{Q}}, W') \otimes \mathbb{C}$$

then $\tilde{f} : (\tilde{K}, \tilde{W}) \rightarrow (\tilde{K}', \tilde{W}')$ is a morphism of bifiltered complexes such that squares I and II in the diagram below

$$\begin{array}{ccccc} (K_{\mathbb{C}}, W) & \xleftarrow{\alpha_0} & (\tilde{K}, \tilde{W}) & \xrightarrow{\alpha_1} & (K_{\mathbb{Q}}, W) \otimes \mathbb{C} \\ f_{\mathbb{C}} \downarrow & & \text{I} & \tilde{f} \downarrow & \text{II} & \downarrow f_{\mathbb{Q}} \otimes \mathbb{C} \\ (K'_{\mathbb{C}}, W') & \xleftarrow{\alpha'_0} & (\tilde{K}', \tilde{W}') & \xrightarrow{\alpha'_1} & (K'_{\mathbb{Q}}, W') \otimes \mathbb{C} \end{array}$$

commute. In this way we obtain the category $\mathcal{H}dg$ of mixed Hodge complexes.

Next we endow $\mathcal{H}dg$ with a structure of cosimplicial descent category.

Simple functor: If $K = ((K_{\mathbb{Q}}, W), (K_{\mathbb{C}}, W, F), \alpha)$ is a cosimplicial mixed Hodge complex, let $\mathbf{s}_{\mathcal{H}dg}K$ be the mixed Hodge complex $((\mathbf{s}K_{\mathbb{Q}}, \delta W), (\mathbf{s}K_{\mathbb{C}}, \delta W, \mathbf{s}F), \mathbf{s}\alpha)$, where \mathbf{s} denotes the usual simple of cochain complexes and δW is the diagonal filtration. More concretely

$$\begin{aligned} \mathbf{s}K_*^n &= \bigoplus_{p+q=n} K_*^{p,q}; \quad (\delta W)_k(\mathbf{s}K_*)^n = \bigoplus_{i+j=n} W_{k+i}K_*^{i,j}, \quad \text{if } * \text{ is } \mathbb{Q} \text{ or } \mathbb{C} \\ (\mathbf{s}F)^k(\mathbf{s}K_{\mathbb{C}})^n &= \bigoplus_{p+q=n} F^k K_{\mathbb{C}}^{p,q}. \end{aligned}$$

If $\alpha = (\alpha_0, \alpha_1, (\tilde{K}, \tilde{W}))$ then $\mathbf{s}\alpha$ denotes the zig-zag

$$(\mathbf{s}K_{\mathbb{C}}, \delta W) \xleftarrow{\mathbf{s}\alpha_0} (\mathbf{s}\tilde{K}, \delta \tilde{W}) \xrightarrow{\mathbf{s}\alpha_1} (\mathbf{s}(K_{\mathbb{Q}} \otimes \mathbb{C}), \delta(W \otimes \mathbb{C})) \simeq (\mathbf{s}K_{\mathbb{Q}}, \delta W) \otimes \mathbb{C}$$

Equivalences: Set $E_{\mathcal{H}dg} = \{f = (f_{\mathbb{Q}}, f_{\mathbb{C}}, \tilde{f}) \mid f_{\mathbb{Q}} \text{ is a quasi-isomorphism in } Ch^+\mathbb{Q}\}$. It follows from general Hodge theory that an equivalence $f : K \rightarrow K'$ induces isomorphism on the resulting Hodge structures.

Transformation λ : $\lambda^{\mathcal{H}dg} : Id_{\mathcal{H}dg} \rightarrow \mathbf{s}_{\mathcal{H}dg}(- \times \Delta)$ is $\lambda_K^{\mathcal{H}dg} = (\lambda_{K_{\mathbb{Q}}}^{\mathbb{Q}}, \lambda_{K_{\mathbb{C}}}^{\mathbb{C}}, \lambda_K^{\mathbb{C}})$ induced by the transformations $\lambda^{\mathbb{Q}}$ and $\lambda^{\mathbb{C}}$ of $Ch^+\mathbb{Q}$ and $Ch^+\mathbb{C}$ respectively.

Transformation μ : Similarly, $\mu_Z^{\mathcal{H}dg} = (\mu_{Z_{\mathbb{Q}}}^{\mathbb{Q}}, \mu_{Z_{\mathbb{C}}}^{\mathbb{C}}, \mu_Z^{\mathbb{C}}) : \mathbf{s}_{\mathcal{H}dg}\mathbf{s}_{\mathcal{H}dg}Z \rightarrow \mathbf{s}_{\mathcal{H}dg}DZ$.

PROPOSITION 5.13. *The category $(\mathcal{H}dg, \mathbf{s}_{\mathcal{H}dg}, E_{\mathcal{H}dg}, \mu^{\mathcal{H}dg}, \lambda^{\mathcal{H}dg})$ is an additive cosimplicial descent category. In addition, the forgetful functor $U : \mathcal{H}dg \rightarrow Ch^+\mathbb{Q}$ given by $U((K_{\mathbb{Q}}, W), (K_{\mathbb{C}}, W, F), \alpha) = K_{\mathbb{Q}}$ is a functor of additive cosimplicial descent categories.*

Proof. Note that $\mathbf{s}_{\mathcal{H}dg} = (\mathbf{s}, \delta, \mathbf{s}) : \Delta\mathcal{H}dg \rightarrow \mathcal{H}dg$ is indeed a functor. Given $K \in \Delta\mathcal{H}dg$, then $\mathbf{s}_{\mathcal{H}dg}K$ is a mixed Hodge complex by [DeIII, 8.I.15 i], and $\mathbf{s}_{\mathcal{H}dg}$ is functorial with respect to the morphisms of $\Delta\mathcal{H}dg$ by definition.

The us see that transfer lemma is satisfied for $U : \mathcal{H}dg \rightarrow Ch^+\mathbb{Q}$. To see (SDC 4)'^{op} and (SDC 5)'^{op}, let $K = ((K_{\mathbb{Q}}, W), (K_{\mathbb{C}}, W, F), \alpha)$ be a mixed Hodge complex. Clearly $\lambda_{K_{\mathbb{Q}}}^{\mathbb{Q}}, \lambda_{K_{\mathbb{C}}}^{\mathbb{C}}$ and $\lambda_K^{\mathbb{C}}$ preserve the filtrations. Set $K = K \times \Delta$. As the following diagram commutes in $CF^+\mathbb{C}$

$$\begin{array}{ccccccc} (K_{\mathbb{C}}, W) & \xleftarrow{\alpha_0} & (\tilde{K}, \tilde{W}) & \xrightarrow{\alpha_1} & (K_{\mathbb{Q}}, W) \otimes \mathbb{C} & & \\ \lambda_{K_{\mathbb{C}}}^{\mathbb{C}} \downarrow & & \lambda_{\tilde{K}}^{\mathbb{C}} \downarrow & & \lambda_{K_{\mathbb{Q}} \otimes \mathbb{C}}^{\mathbb{C}} \downarrow & \searrow \lambda_{K_{\mathbb{Q}}}^{\mathbb{Q}} \otimes \mathbb{C} & \\ (\mathbf{s}K_{\mathbb{C}}, \delta(W)) & \xleftarrow{\mathbf{s}(\alpha_0)} & (\mathbf{s}\tilde{K}, \delta(\tilde{W})) & \xrightarrow{\mathbf{s}(\alpha_1)} & (\mathbf{s}(K_{\mathbb{Q}} \otimes \mathbb{C}), \delta(W \otimes \mathbb{C})) & \xrightarrow{\sim} & (\mathbf{s}K_{\mathbb{Q}}, \delta(W)) \otimes \mathbb{C} \end{array}$$

then $\lambda_K^{\mathcal{H}dg} = (\lambda_{K_Q}^Q, \lambda_{K_C}^C, \lambda_{\tilde{K}}^{\tilde{C}})$ is a morphism in $\mathcal{H}dg$. It can be argued similarly with $\mu_K^{\mathcal{H}dg} = (\mu_{K_Q}^Q, \mu_{K_C}^C, \mu_{\tilde{K}}^{\tilde{C}})$, and with the quasi-inverse $\rho_K^{\mathcal{H}dg}$ of $\lambda_K^{\mathcal{H}dg}$. On the other hand, (FD 1)^{op} is trivial since U is additive, and diagram

$$\begin{array}{ccc} \Delta\mathcal{H}dg & \xrightarrow{U} & \Delta Ch^+\mathbb{Q} \\ \downarrow (s, \delta, s) & & \downarrow s \\ \mathcal{H}dg & \xrightarrow{U} & Ch^+\mathbb{Q} \end{array}$$

is clearly commutative. Also, the transformations λ of $\mathcal{H}dg$ and $Ch^+\mathbb{Q}$ are compatible, and the same happens with μ , so lemma 3.3^{op} holds. \square

Fibrant Spectra

We describe now a cosimplicial descent structure on fibrant spectra. Essentially all axioms of cosimplicial descent category for fibrant spectra are proved in [T, section 5], taking the homotopy limit as simple functor, and the stable equivalences as equivalences. Indeed, the proofs in loc. cit. are based in the properties of the homotopy limit of cosimplicial simplicial sets developed in [BK].

Recall that a *fibrant spectrum* X is a family of fibrant pointed simplicial sets $\{X_n\}_{n \in \mathbb{N}}$ together with maps $S^1 \wedge X_n \rightarrow X_{n+1}$ that are weak (homotopy) equivalences of simplicial sets. Morphisms of spectra are defined in the obvious way. We denote by Sp the category of fibrant spectra.

In this case the ‘stable equivalences’, i.e., the morphisms of spectra inducing isomorphism on the stable homotopy groups, are the same as the degree-wise weak homotopy equivalences. Then, set

$$E = \{f : X \rightarrow Y \text{ such that } f_n \text{ is a weak (homotopy) equivalence in } \Delta^\circ Set_* \text{ for all } n\}$$

The simple functor is the homotopy limit, $\text{holim} : \Delta Sp \rightarrow Sp$, given degree-wise as the homotopy limit of (pointed) cosimplicial simplicial sets [BK], $(\text{holim} X)_n = \text{holim}(X_n)$, that is defined as follows.

The ‘function space’ $\text{hom}_* : (\Delta\Delta^\circ Set_*)^{\text{op}} \times \Delta\Delta^\circ Set_* \rightarrow \Delta^\circ Set_*$ is in degree n

$$\text{hom}_n(S, T) = \text{Hom}_{\Delta\Delta^\circ Set_*}(S \times \Delta[n], T)$$

Define $\Delta/- \in \Delta\Delta^\circ Set_*$ as $N(\Delta/[n])$ in cosimplicial degree n . $N(\Delta/[n])$ denotes the nerve of the category $\Delta/[n]$. If $Z \in \Delta\Delta^\circ Set_*$, then $\text{holim} Z = \text{hom}_*(\Delta/[-], Z)$. If we replace Δ by an arbitrary small category I , we get a 2-functor $\text{holim}_I : ISp \rightarrow Sp$ [BK, T].

PROPOSITION 5.14. *The category Sp together with holim as simple functor and E as equivalences is a cosimplicial descent category.*

Proof. Axioms (S1)^{op} and (S2)^{op} are basic properties of Sp inherited from $\Delta^\circ Set_*$. (S8)^{op} also holds since $\Upsilon : \Delta \rightarrow \Delta$ is an isomorphism and holim a 2-functor, so $\text{holim} \Upsilon \simeq \text{holim}$. Being holim a 2-functor, then compatible natural transformations λ and μ are easily defined. Indeed, λ is obtained from $\Delta \rightarrow *$, and a choice of $* \rightarrow \Delta$ provides ρ with $\rho\lambda = Id$. On the other hand, $d : \Delta \rightarrow \Delta \times \Delta$ provides the transformation μ , since the Fubini property of holim ensures that $\text{holim}_{\Delta \times \Delta}$ and holim_Δ applied twice are the same.

Now, axioms (S3)^{op}, (S4)^{op} and (S6)^{op} are exactly lemmas 5.11, 5.33 and 5.8 in [T]. (S5)^{op} follows from lemma 5.25 in loc. cit.. It states that the natural map $\text{hom}_*(\Delta/[-], X) \rightarrow \text{holim} X$ is in E when X is a so called ‘fibrant cosimplicial fibrant spectrum’, where $\Delta/[-] \in \Delta\Delta^\circ Set_*$ is $\Delta/[-]^n = \Delta[n]$. But $X = Z \times \Delta$ is always fibrant in the previous sense for $Z \in Sp$. Then, we get the weak equivalence $Z = \text{hom}_*(\Delta/[-], Z \times \Delta) \rightarrow \text{holim}(Z \times \Delta)$, that is just λ_Z .

To finish it remains to prove (S7)^{op}. We remark that, although not the same, it is closely related to [T, lemma 5.12]. Consider a morphism $f : X \rightarrow Y$ in Sp . It defines the ‘homotopy fibration sequence’

$$\Omega X \rightarrow Ff \rightarrow X \rightarrow Y \quad (6)$$

defined as follows. Denote by Map_* the internal hom in Sp (inherited from the one in $\Delta^\circ Set_*$). Then $\Omega X = Maps_*(S^1, X)$ and Ff is the pullback of $Maps_*(\Delta[1], Y) \rightarrow Y \wedge Y$ and $f \wedge * : X \wedge X \rightarrow Y \wedge Y$. The sequence (6) induces the classical long exact sequence on the (stable) homotopy groups

$$\cdots \rightarrow \pi_{k+1}X \rightarrow \pi_k Ff \rightarrow \pi_k X \rightarrow \pi_k Y \rightarrow \pi_{k-1} Ff \rightarrow \cdots$$

Then $f \in E$ if and only if $\pi_k Ff = 0$ for all $k \in \mathbb{Z}$. We state that Ff and the homotopy limit of the cosimplicial fibre of $f \times \Delta$ are weak equivalent. In this case we would be done.

Since all ingredients in the previous statement are defined degree-wise, we just need to check it for $f : X \rightarrow Y$ in $\Delta^\circ Set_*$. Recall that holim commutes with pullbacks, and E is closed under them. Therefore we only need to see that $\text{holim}(Y^{\Delta[1]})$ and $Map_*(\Delta[1], Y)$ are weakly equivalent. Now, $Y^{\Delta[1]} \in \Delta\Delta^\circ Set_*$ is defined as $(Y^{\Delta[1]})^n = \wedge^{\Delta[1]^n} Y$, and $-\Delta[1]$ is right adjoint to $\Delta[1] \wedge -$, where $(\Delta[1] \wedge Z)^n = \bigvee_{\Delta[1]^n} Z^n$. Therefore, if $Z \in \Delta\Delta^\circ Set_*$, then $\text{Hom}_{\Delta\Delta^\circ Set_*}(Z, Y^{\Delta[1]}) \simeq \text{Hom}_{\Delta\Delta^\circ Set_*}(\Delta[1] \wedge Z, Y)$. Hence $\text{hom}_*(\Delta/-, Y^{\Delta[1]}) \simeq \text{hom}_*(\Delta[1] \wedge \Delta/-, Y)$. As $\Delta[1]$ is the nerve of $I = \{\bullet \rightarrow \bullet\}$, and the nerve commutes with products, then

$$\text{holim}(Y^{\Delta[1]}) \simeq \text{hom}_*((I \times \Delta)/-, Y) = \text{holim}_{I \times \Delta} Y = \text{holim}_I(\text{holim} Y)$$

But $\lambda_Y : Y \rightarrow \text{holim} Y$ is a weak equivalence, and $Map_*(\Delta[1], Y) \simeq \text{holim}_I Y$, so we get the desired weak equivalence $Map_*(\Delta[1], Y) \rightarrow \text{holim}(Y^{\Delta[1]})$. \square

6 Derived functors in simplicial descent categories

Recall that the Godement resolution $\mathcal{C}_{\text{God}}\mathcal{F}$ of a sheaf of chain complexes \mathcal{F} over X provides a cosimplicial sheaf complex that can be used to define the usual sheaf hypercohomology as $\mathbb{H}(X, \mathcal{F}) = \mathbf{s}\Gamma(X, \mathcal{C}_{\text{God}}\mathcal{F})$.

If \mathcal{F} is now a sheaf of fibrant spectra, then the ‘sheaf hypercohomology spectrum’ of X with coefficients in F [T, 1.33] is by definition $\mathbb{H}(X, \mathcal{F}) = \text{holim}\Gamma(X, \mathcal{C}_{\text{God}}\mathcal{F})$.

Also, if $X \rightarrow Y$ is a continuous map and \mathcal{F} a sheaf of commutative differential graded algebras (over a field of zero characteristic), then Navarro’s ‘Thom-Whitney’ derived functor $\mathbb{R}f_*$ is defined as $\mathbb{R}f_*\mathcal{F} = \mathbf{s}_{TW}f_*\mathcal{C}_{\text{God}}\mathcal{F}$.

Note that the cosimplicial Godement resolution is a particular case of a cosimplicial object coming from a triple. More concretely, it is constructed from the triple associated to the adjoint pair (p^*, p_*) , induced by $p : X_{\text{dis}} \rightarrow X$. Here X_{dis} is the underlying set of X with the discrete topology.

Recall that the category $PrSh(X, \mathcal{D})$ of presheaves over X with values in a cosimplicial descent category \mathcal{D} is again a cosimplicial descent category by lemma 2.6. If the simple functor $\mathbf{s} : \Delta PrSh(X, \mathcal{D}) \rightarrow PrSh(X, \mathcal{D})$ preserves sheaves, then the category $Sh(X, \mathcal{D})$ of sheaves over X with values in \mathcal{D} is also a cosimplicial descent category. In many cases \mathbf{s} is defined by an end, in particular it is given by a limit, and it is not difficult to check that \mathbf{s} preserves sheaves.

Thus, previous examples are just the right derived functors $\mathbb{R}F$ of functors $F : \mathcal{D} \rightarrow \mathcal{E}$, where \mathcal{D} is a cosimplicial descent category. $\mathbb{R}F$ is computed as the simple functor of a cosimplicial resolution obtained through a triple.

Next we formalize the construction of right derived functors on cosimplicial descent categories. The dual procedure will produce left derived functors on simplicial descent categories. We make use of the Cartan-Eilenberg categories machinery [GNPR] to derive functors.

DEFINITION 6.1. [GNPR] Consider the data $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ where $\mathcal{S} \subseteq \mathcal{W}$ are two classes of distinguished morphisms in the category \mathcal{C} . They are called respectively *strong* and *weak equivalences*.

An object M is said to be fibrant if for each $w : X \rightarrow Y \in \mathcal{W}$, then $w^* : \text{Hom}_{\mathcal{C}[\mathcal{S}^{-1}]}(Y, M) \rightarrow \text{Hom}_{\mathcal{C}[\mathcal{S}^{-1}]}(X, M)$, $g \mapsto gw$ is a bijection. A Cartan-Eilenberg category is $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ as before and such that

If $X \in \mathcal{C}$, there is a *fibrant* object M and a weak equivalence $X \rightarrow M$.

In this case, the full subcategory $\mathcal{C}_{\text{fib}}[\mathcal{S}^{-1}, \mathcal{C}]$ of $\mathcal{C}[\mathcal{S}^{-1}]$ of fibrant objects is equivalent to $\mathcal{C}[\mathcal{W}^{-1}]$.

In loc. cit. [GNPR, theorem 3.2.1] it is proven that a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ such that $F(s)$ is an isomorphism for every $s \in \mathcal{S}$, has a right derived functor $\mathbb{R}F : \mathcal{C}[\mathcal{W}^{-1}] \rightarrow \mathcal{D}$ in the sense of Quillen [Q].

Next we will see that a simplicial descent category with a compatible triple provides a Cartan-Eilenberg structure. Let us fix some notations.

Recall that a triple $\mathbf{T} = (T, \eta, \nu)$ on \mathcal{D} is a functor $T : \mathcal{D} \rightarrow \mathcal{D}$ together with natural transformations $\eta : \text{Id}_{\mathcal{D}} \rightarrow T$ and $\nu : T^2 \rightarrow T$. They must satisfy the usual compatibility properties [ML]. If $X \in \mathcal{D}$, the cosimplicial object $\overline{T}(X)$ is defined as $\overline{T}^n X = T^{n+1} X$, with face and degeneracy maps given by η and ν . Note that η provides a coaugmentation $\overline{\eta}_X : X \rightarrow \overline{T}X$.

$$\begin{array}{ccccccc}
 X & \xrightarrow{\eta_X} & TX & \xrightleftharpoons[\eta_X]{\eta_{TX}} & T^2X & \xrightleftharpoons[\eta_X]{\eta_{T^2X}} & T^3X \rightleftarrows T^4X \dots\dots
 \end{array}$$

ν_X (curved arrow from TX to T^2X)
 ν_{TX} (curved arrow from T^2X to T^3X)
 $T\nu_X$ (curved arrow from TX to T^3X)

DEFINITION 6.2. A triple \mathbf{T} is *compatible with the descent structure* $(\mathcal{D}, \mathbf{E}, \mu, \lambda)$ on \mathcal{D} if $T(\mathbf{E}) \subset \mathbf{E}$ and the following condition holds.

There is a natural transformation $\theta : T\mathbf{s} \rightarrow \mathbf{s}T$ in \mathbf{E} , such that

1. $\theta_Y \eta_{\mathbf{s}Y} = \mathbf{s} \eta_Y$.
2. $\mathbf{s} \nu_Y \theta_Y^2 = \theta_Y \nu_{\mathbf{s}Y}$ for all $Y \in \Delta \mathcal{D}$, where θ^2 is the iteration of θ given below.

The iterates $\theta^i : T^i \mathbf{s} \rightarrow \mathbf{s} T^i$ of θ are the natural transformations defined as $\theta_X^i = \theta_{T^{i-1}X} T(\theta_X^{i-1})$, if $X \in \text{Ob} \mathcal{D}$. Note that θ_X^i is also in \mathbf{E} .

THEOREM 6.3. Let $(\mathcal{D}, \mathbf{E}, \mathbf{s}, \mu, \lambda)$ be a cosimplicial descent category and \mathbf{T} a compatible triple. Set $\mathcal{S} = \mathbf{E}$, $\mathbf{F} = \mathbf{s}\overline{T}$ and $\mathcal{W} = \mathbf{F}^{-1}(\mathbf{E})$. Then $(\mathcal{D}, \mathcal{S}, \mathcal{W})$ is a right Cartan-Eilenberg category. The fibrant objects are those isomorphic to $\mathbf{F}X$ for some object X in \mathcal{D} .

The above result generalizes [GNPR, theorem 5.1.5], where \mathcal{D} is the category of positive complexes over an additive category.

LEMMA 6.4. Let $(\mathcal{D}, \mathbf{s}, \mathbf{E}, \mu, \lambda)$ be a cosimplicial descent category. If $\alpha : X_{-1} \times \Delta \rightarrow X$ is a coaugmentation with an extra degeneracy then $\mathbf{s}\alpha \in \mathbf{E}$.

Proof. Assume there is an extra degeneracy $s^{-1} : X_n \rightarrow X_{n-1}$ (or s^n), $n \geq 0$, satisfying the simplicial identities. This implies that there exists $\beta : X \rightarrow X_{-1} \times \Delta$ with $\beta\alpha = \text{Id}_{X_{-1}}$ and with $\alpha\beta$ homotopic to Id_X (see, for instance [B, chap. 3, 3.2]). Then, there is a homotopy $H : \text{Cyl}(X) \rightarrow X_{-1}$ with $Hd^0 = \text{Id}_X$ and $Hd^1 = \alpha\beta$ (see [May, prop. 6.2]). By proposition 2.9, sd^0 and sd^1 are equivalences. Then, $\mathbf{s}H \in \mathbf{E}$, and $\mathbf{s}\alpha$, $\mathbf{s}\beta$ are isomorphisms in $\text{Ho}\mathcal{D}$. But \mathbf{E} is saturated, so $\mathbf{s}\alpha \in \mathbf{E}$. \square

Proof of theorem 6.3. Consider the natural transformation $\epsilon : Id_{\mathcal{D}} \rightarrow F$, where $\epsilon_X : X \rightarrow FX$ is the composition

$$X \xrightarrow{\lambda_X} s(X \times \Delta) \xrightarrow{s\bar{\eta}_X} FX = s\bar{T}X \quad .$$

By [GNPR, thm. 2.5.4], it suffices to verify that

$$F(\mathcal{S}) \subseteq \mathcal{S} \quad , \quad F\epsilon_X \in \mathcal{S} \quad \text{and} \quad \epsilon_{FX} \in \mathcal{S}, \text{ for every } X \in \mathcal{D} \quad . \quad (7)$$

If $f \in \mathcal{S} = E$, then $\bar{T}^n f = T^{n+1} f \in E$. Hence, $Ff = s(\{\bar{T}^n f\}_n)$ is also in \mathcal{S} by (S6). Let us check that $F\epsilon_X$, $\epsilon_{FX} \in \mathcal{S}$ for every X in \mathcal{D} . By (S5), $\lambda_X, \lambda_{FX} \in \mathcal{S}$. As F preserves equivalences then $F\lambda_X \in \mathcal{S}$. Let us see that $Fs\bar{\eta}_X$, $s\bar{\eta}_{FX}$ are in \mathcal{S} .

We have that $Fs\bar{\eta}_X = s\bar{T}s\bar{\eta}_X$, which is the simple of the cosimplicial morphism $n \rightarrow T^{n+1}s\bar{\eta}_X$. By (S6), it suffices to prove that $T^{n+1}s\bar{\eta}_X$ is an equivalence for each $n \geq 0$. Note that $\theta^i : T^i s \rightarrow sT^i$ is an object-wise equivalence. Then, $T^{n+1}s\bar{\eta}_X$ is in E if and only if $sT^{n+1}\bar{\eta}_X$ is. But the coaugmentation $T^{n+1}\bar{\eta}_X$ has clearly an extra degeneracy, so it follows from lemma 6.4 that $sT^{n+1}\bar{\eta}_X \in E$. Hence, $Fs\bar{\eta}_X \in E$.

On the other hand, $s\bar{\eta}_{FX}$ is the image under s of $\bar{\eta}_{s\bar{T}X} : s\bar{T}X \times \Delta \rightarrow \bar{T}s\bar{T}X$. Denote $s\bar{T}\bar{T}X = s(n \rightarrow T^{n+1}\bar{T}X)$. The transformations $\tau^m = \theta_{\bar{T}X}^{m+1} : T^{m+1}s\bar{T}X \rightarrow sT^{m+1}\bar{T}X$ provide a natural morphism $\tau_X : \bar{T}s\bar{T}X \rightarrow s\bar{T}\bar{T}X$. By definition, τ_X is a degree-wise equivalence, and it is a morphism of cosimplicial objects since T is compatible with the descent structure. Also,

$$\tau_X \bar{\eta}_{s\bar{T}X} = s(k \rightarrow \bar{\eta}_{\bar{T}^k X}) : (s\bar{T}X) \times \Delta \rightarrow s\bar{T}\bar{T}X \quad (8)$$

Hence, if we apply s to (8) we deduce that $s(\bar{\eta}_{FX}) \in \mathcal{S}$ if and only if $s(m \rightarrow s(n \rightarrow \bar{\eta}_{T^{n+1}X}^m)) \in \mathcal{S}$. By (S4), this is equivalent to prove that $s(n \rightarrow s(m \rightarrow \bar{\eta}_{T^{n+1}X}^m)) \in \mathcal{S}$. Since $\bar{\eta}_{T^{n+1}X}$ has again an extra degeneracy for all $n \geq 0$, then $s(m \rightarrow \bar{\eta}_{T^{n+1}X}^m) \in \mathcal{S}$, and by (S6), $s(n \rightarrow s(m \rightarrow \bar{\eta}_{T^{n+1}X}^m)) \in \mathcal{S}$ as required. \square

Under the hypothesis of the previous theorem, (F, ϵ) is a so called ‘right resolvent functor’ [GNPR], so the fibrant objects are those isomorphic to FX for some $X \in \mathcal{D}$.

COROLLARY 6.5. *Let $(\mathcal{D}, E, s, \mu, \lambda)$ be a cosimplicial descent category and \mathbf{T} a compatible triple. Consider a functor $F : \mathcal{D} \rightarrow \mathcal{E}$ such that Fs is an isomorphism for every $s \in E$. Then there exists a right derived functor $\mathbf{R}F : \mathcal{D}[\mathcal{W}^{-1}] \rightarrow \mathcal{E}$, which is a left Kan extension of F . It can be computed as $\mathbf{R}FX = Fs\bar{T}X$.*

A deeper study of derived functors in (co)simplicial descent categories -with emphasis on sheaf cohomology of algebras over an operad- will appear in a joint work with A. Roig.

References

- [B] M. Barr, *Acyclic models*, CRM Monograph Series, **17**. Amer. Math. Soc., Providence, 2002.
- [Be] A. A. Beilinson, *Notes on absolute Hodge cohomology. Applications of algebraic K-theory to algebraic geometry and number theory*, Contemp. Math., **55**, Amer. Math. Soc., Providence, RI, (1986) p. 35-68.
- [BG] A. K. Bousfield and A. M. Gugenheim, *On PL De Rham theory and rational homotopy type*, Mem. Amer. Math. Soc. **179** (1976).
- [BK] A. K. Bousfield, D. M. Kan, *Homotopy limits, completions and localizations*, Lect. Notes in Math. **304**, Springer, Berlin (1972).

- [DeII] P. Deligne, *Théorie de Hodge II*, Publ. Math. I.H.E.S., **40** (1971), p. 5-57.
- [DeIII] P. Deligne, *Théorie de Hodge III*, Publ. Math. I.H.E.S., **44** (1975), p. 2-77.
- [DP] A. Dold and D. Puppe, *Homologie Nicht-Additiver Funktoren. Anwendungen*, Ann. Inst. Fourier, **11** (1961), p. 201-312.
- [Dup] J. Dupont, *Curvature and Characteristic Classes*, Lect. Notes in Math. **640**, Springer, Berlin (1978).
- [EM] S. Eilenberg and J.C. Moore, *Homology and fibrations. I. Coalgebras, cotensor product and its derived functors*, Comment. Math. Helv. **40** (1966), p. 199-236.
- [GM] S.I. Gelfand and Y.I. Manin, *Methods in Homological Algebra*, Second ed., Springer Monographs in Math., Berlin, 2003.
- [GN] F. Guillén and V. Navarro Aznar, *Un Critère d'Extension des Foncteurs Définis sur les Schémas Lisses*, Publ. Math. I.H.E.S., **95** (2002), 1-91.
- [GNPR] F. Guillén Santos, V. Navarro, P. Pascual and A. Roig, *A Cartan-Eilenberg approach to Homotopical Algebra*, Journal of Pure and Applied Algebra (2009), doi:10.1016/j.jpaa.2009.04.009.
- [H] A. Huber, *Mixed Motives and their Realization in Derived Categories*, Lect. Notes in Math., **1604**, Springer, Berlin (1995).
- [III] L. Illusie, *Complexe Cotangent et Déformations I*, Lect. Notes in Math., **239**, Springer, Berlin (1971).
- [K] B. Keller, *Deriving DG categories*, Ann. Sci. École Norm. Sup. **27** (1994), n. 1, p. 63-102.
- [ML] S. MacLane, *Categories for the Working Mathematician*, Second ed., Springer, Berlin, 1998.
- [M] J. P. May, *The geometry of iterated loop spaces*, Lect. Notes in Math., **271**, Springer, Berlin (1972).
- [May] J. P. May, *Simplicial Objects in Algebraic Topology*, Van Nostrand, Princeton, 1967.
- [N] V. Navarro Aznar, *Sur la théorie de Hodge-Deligne*, Invent. math., **90** (1987), p. 11-76.
- [PS] Peters, C. and Steenbrink, J., *Mixed Hodge structures*, Ergebnisse der Math.; Ser. of Modern Surveys in Mathematics, **52**, Springer-Verlag, Berlin (2008).
- [Q] D. Quillen, *Homotopical Algebra*, Lect. Notes in Math., **43**, Springer, Berlin (1967).
- [R] B. Rodríguez-González, *Categorías de Descenso Simplicial*, PhD thesis, University of Seville, 2007. Translated and revised version at arXiv:0804.2154v1.
- [R1] B. Rodríguez-González, *Triangulated structures induced by simplicial descent categories*, preprint 2009.
- [S] G. Segal, *Categories and cohomology theories*, Topology **13**, (1974), p. 293-312.
- [T] R. W. Thomason, *Algebraic K-theory and etale cohomology*, Ann. Sci. de L'E.N.S **18**, (1985), p. 437-552.
- [Vo] V. Voevodsky, *Simplicial additive functors* preprint, 2007, arXiv:0805.4434v1.